B.A./B.Sc. 2nd Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

Paper : BMH2CC03

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols bear their usual meaning.

1. Answer *any ten* questions from the following: 2×10=20

(a) Find Sup A and Inf A for $A = \left\{\frac{n+(-1)^n}{n} : n \in \mathbb{N}\right\}$.

(b) If S be an enumerable subset and T be an uncountably infinite subset of $\mathbb{R}$. Prove that T-S is uncountable.

(c) Give an example to show that $(A \cap B)'$ may not be equal to $A' \cap B'$, where A' denotes the derived set of A.

(d) Show that -1 is a limit point of the set $S = \left\{(-1)^n\left(1+\frac{1}{n}\right) : n \in \mathbb{N}\right\}$.

(e) Show that for a closed set F, $F' \subset F$.

(f) Let G be an open set in $\mathbb{R}$ and S be a subset of $\mathbb{R}$ such that $G \cap S = \varphi$. Prove that $G \cap S' = \varphi$.

(g) Give an example to show that arbitrary intersection of open sets in R may not be open.

(h) Find $\lim_{n\to\infty} \sup u_n$ and $\lim_{n\to\infty} \inf u_n$, where $u_n = n^{(-1)^n}$.

(i) If $p > 0$, then find $\lim_{n\to\infty} p^{\frac{1}{n}}$.

(j) Prove that $\lim_{n\to\infty} \frac{\{(n+1)(n+2)\ldots(2n)\}^{\frac{1}{n}}}{n} = \frac{4}{e}$.

(k) Prove that $\lim_{n\to\infty} \frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\cdots+n^{\frac{1}{n}}}{n} = 1$.

(l) Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1} n^{-\frac{1}{2}}$ is convergent. Examine its absolute convergency.

(m) Show that $\sum \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$ is convergent.

(n) Show that the series $\frac{1}{\log 2} + \frac{1}{\log 3} + \frac{1}{\log 4} + \cdots$ diverges.

(o) Test the convergence of $\sum_{n=1}^{\infty} \cos\frac{1}{n^2}$.

5×4=20

2. Answer *any four* questions :

(a) State Archimedean property of $\mathbb{R}$. If $x \in \mathbb{R}$ with $x > 0$, then show that there is a natural number m such that $m - 1 \leq x < m$. Is the result true for $x \leq 0$? Give reason. 1+2+2=5

(b) (i) A sequence $\{x_n\}$ is defined as follows:

$$x_1 \leq x_3 \leq x_5 \leq \cdots \leq x_6 \leq x_4 \leq x_2$$

If $(x_{2n} - x_{2n-1}) \to 0$ as $n \to \infty$, show that $\{x_n\}$ is convergent.

(ii) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $|x_{n+1} - x_n| < \frac{1}{n^2}, \forall\, n \in \mathbb{N}$. Show that $\{x_n\}$ is convergent. 2+3=5

(c) (i) Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = l \in [0, 1)$, show that $\lim\limits_{n\to\infty} u_n = 0$.

(ii) Show that a monotone increasing sequence which is bounded above is convergent. 3+2=5

(d) (i) Show that the series $1 - \frac{(2!)^2}{4!} + \frac{(3!)^2}{6!} - \cdots$ converges absolutely.

(ii) If $\sum\limits_{n=1}^{\infty} y_n$ be a convergent series of positive reals, prove that $\sum\limits_{n=1}^{\infty} y_n^{\,2}$ is convergent. 3+2=5

(e) (i) Show that the series $\left(\frac{1}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2 + \cdots$ is divergent.

(ii) Let $x_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$ for all $n \in \mathbb{N}$. Test whether $\{x_n\}$ is a cauchy sequence or not 2+3=5

(f) Define countable set. Prove that countable union of countable sets is countable. 1+4=5

3. Answer *any two* questions: 10×2=20

(a) (i) State and prove D′Alembert's Ratio Test.

(ii) Show that the series

$\frac{3}{2} - \frac{4}{3} + \frac{5}{4} - \frac{6}{5} + \cdots$ is not convergent but the series $\left(\frac{3}{2} - \frac{4}{3}\right) + \left(\frac{5}{4} - \frac{6}{5}\right) + \cdots$ is convergent. Give reasons for obtaining different nature of convergence of the series before and after grouping of terms. (1+3)+(2+2+2)=10

(b) (i) Prove that, the least upper bound of a bounded set, if it does not belongs to the set, is the limit point of the set.

(ii) Show that the collection $H = \left\{\left(\frac{1}{n}, \frac{2}{n}\right) : n = 2, 3, \ldots\right\}$ of open intervals is an open covering of the open interval (0, 1). Examine if any finite sub-collection of H can cover the interval (0, 1). Is the closed interval [0, 1] compact? Justify your answer. 4+(3+2+1)=10

(c) (i) If $\{x_n\}$ and $\{y_n\}$ are two real sequences such that $\lim_n x_n = l$ and $\lim_n y_n = m\ (\neq 0)$, prove that $\lim_n (x_n/y_n) = l/m$.

(ii) Two sequences $\{x_n\}$, $\{y_n\}$ are defined by $x_{n+1} = \frac{1}{2}(x_n + y_n), y_{n+1} = \sqrt{x_n y_n}$ for $n \geq 1$ and $x_1 > 0,\ y_1 > 0$. Prove that both the sequences converge to a common limit.

(iii) If $\lim_{n\to\infty} x_n = l$, then show that $\lim_{n\to\infty} |x_n| = |l|$. Give an example to show that the converse is not true.

4+4+2=10

(d) (i) Test for convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$.

(ii) Test the convergence of the series:

(I) $\sin\frac{\pi}{2} + \sin\frac{\pi}{4} + \sin\frac{\pi}{6} + \cdots$

(II) $\sum_{n=1}^{\infty} \sin\frac{\pi}{2n^2}$

(iii) If $\sum_{n=1}^{\infty} x_n$ is a convergent series of positive terms then examine if the series $\sum_{n=1}^{\infty} x_n^2$ is convergent.

4+(2+2)+2=10

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