B.A./B.Sc 4th Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

Paper : BMH4 CC10

(Ring Theory and Linear Algebra I)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

[Notations and Symbols have their usual meaning.]

Group-A

Marks : 20

1. Answer *any ten* questions: 2×10=20

(a) Show that a ring R is commutative if $x^3 = x$ for all $x \in R$.

(b) What are ideals of a field? Justify your answer.

(c) Suppose R is the ring of all real valued continuous functions defined on the closed interval $[0, 1]$ and let $S = \left\{f \in R : f\left(\frac{1}{2}\right) = 0\right\}$. Then S is an ideal of R. — Justify.

(d) The ring $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$ is a field. — Justify.

(e) Suppose F is a field with 2^n elements, where $n \in N$. Find the characteristic of F.

(f) Define a homomorphism from the ring $\mathbb{Z}$ of integers into the ring $\mathbb{Z}_5$ of integers module 5.

(g) Give an example to show that a quotient ring of an integral domain may not be a field.

(h) Let R be a commutative ring of characteristic 2. Define a map $\varphi: R \to R$ by $\varphi(a) = a^2 \ \forall\, a \in R$. Prove that φ is a ring homomorphism.

(i) Is (0, 0, 1) a linear combination of (1, 0, 1) and (0, 1, 1)? Justify your answer.

(j) If $S = \{(1,0,0), (0,1,0)\}$, describe geometrically the linear span of S in the real vector space $\mathbb{R}^3$.

(k) Is the union of two subspaces of a vector space V a subspace of V? Justify your answer.

(l) Let V be a finite dimensional vector space and W be a subspace of V. What is the relation among dim $V/_W$, dim V and dim W?

(m) Is the map $T(x, y) = (x, y + 3), \forall\, x, y \in \mathbb{R}$ a linear transformation from the real vector space $\mathbb{R}^2$ into itself? Justify your answer.

(n) State the rank-nullity theorem for vector spaces.

(o) It is given that the map $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x,\ x + y,\ y)\ \forall\, x, y \in \mathbb{R}$ is a linear transformation from the real vector space $\mathbb{R}^2$ to the real vector space $\mathbb{R}^3$. Find ker T.

Group-B

Marks : 20

2. Answer *any four* questions: 5×4=20

(a) (i) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$. Prove that $\mathbb{Z}[2]$ is a ring under the usual addition and multiplication of real numbers.

(ii) Let R be a ring and α be a fixed element of R. Show that $I_\alpha = \{x \in R |\ \alpha x = 0\}$ is a subring of R. 3+2=5

(b) (i) Let n be a positive integer. Prove that $n\mathbb{Z}$ is a prime ideal of the ring $\mathbb{Z}$ of integers if and only if n is prime.

(ii) Let R be a ring with unity 1. Prove that R has characteristic $n(\neq 0)$ if and only if n is the smallest positive integer such that $n.1 = 0$. 3+2=5

(c) (i) Let φ be a homomorphism from a ring R onto a ring S. If I is an ideal of R, prove that $\varphi(I)$ is an ideal of S.

(ii) State the third isomorphism theorem for rings. 3+2=5

(d) Let U and W be subspaces of a vector space V over a field F. Prove that

(i) $U + W = \{u + w \mid u \in U,\ w \in W\}$ is a subspace of V.

(ii) $U + W$ is the smallest subspace of V containing U and W. 3+2=5

(e) (i) Find a basis for the real vector space $\mathbb{R}^3$ that contains the vectors (1, 2, 1) and (3, 6, 2).

(ii) It is given that $W = \{(x, y, z) \mid x, y, z \in \mathbb{R}, 2x + y - z = 0\}$ is a subspace of the real vector space $\mathbb{R}^3$. Find the dimension of W. 3+2=5

(f) (i) Let U and V be two finite dimensional vector spaces over the same field F such that dim U = dim V. Show that U and V are isomorphic vector spaces.

(ii) It is given that the map $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y,\ x - y)\ \forall\, x, y \in \mathbb{R}$ is a linear transformation from the real vector space $\mathbb{R}^2$ into itself. Find dim(ImT). 3+2=5

Group-C

Marks : 20

3. Answer *any two* questions: 10×2=20

(a) (i) Let $M_{2\times2}(\mathbb{Z})$ be the ring of all 2×2 matrices over the integers and let $R = \left\{\begin{pmatrix} a & a-b \\ a-b & b \end{pmatrix} \middle| a, b \in \mathbb{Z}\right\}$. Prove or disprove that R is a subring of $M_{2\times2}(\mathbb{Z})$.

(ii) Prove that a finite integral domain is a field.

(iii) Let $\mathbb{R}[x]$ be the ring of polynomials in x with real coefficients and $\langle x^2+1\rangle$ be the principal ideal of $\mathbb{R}[x]$ generated by x^2+1. Prove that $\langle x^2+1\rangle$ is a maximal ideal of $\mathbb{R}[x]$. 2+4+4=10

(b) (i) Let R be a commutative ring with unity and A be an ideal of R. Prove that R/A is a field if and only if A is maximal.

(ii) State and prove the first isomorphism theorem for rings. 5+5=10

(c) (i) Let $S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a+b=0,\ a,b,c,d \in \mathbb{R}\right\}$. Prove that S is a subspace of the vector space $M_{2\times2}(\mathbb{R})$ of all 2×2 real matrices. Find the dimension of S.

(ii) Show that the set of vectors $S = \{(1,2,0),(2,1,3),(1,1,1),(2,3,1)\}$ of vectors is linearly dependent in the real vector space $\mathbb{R}^3$. Find a linearly independent subset T of S such that $L(T) = L(S)$. ($L(A)$ denotes the linear span of A)

(iii) It is given that $U = \{(x,y,z) \in \mathbb{R}^3 | x+2y=z\}$ and $W = \{(x,y,z) \in \mathbb{R}^3 | 2x+2z=y\}$ are subspaces of the real vector space $\mathbb{R}^3$. Find a basis for the subspace $U \cap W$. 5+3+2=10

(d) (i) Let

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

....

....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

be a system of m linear homogenous equations with real coefficients in n variables, where $n > m$. Using Rank-nullity theorem show that the system has a non-trivial solution.

(ii) Let V be a vector space with a basis $\{e^{3t}, te^{3t}, t^2e^{3t}\}$ over the field of real numbers. $D : V \to V$ be defined by $D(f(t)) = \frac{d}{dt}f(t)\ \forall f(t) \in V$. Find the matrix of D in the given basis.

(iii) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^2(\alpha) = \alpha\ \forall \alpha \in \mathbb{R}^2$. 3+5+2=10