B.A./B.Sc 4th Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

Paper : BMH4 CC09

(Multivariate Calculus)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

[Notations and Symbols have their usual meaning.]

Group-A

Marks : 20

1. Answer *any ten* questions: 2×10=20

(a) Let $f(x,y)=\frac{xy}{x^2+y^2}$, for $(x,y)\neq(0,0)$ and $f(0,0)=0$. Find $f_x(0,0)$ and $f_y(0,0)$ and examine for continuity of f at (0, 0).

(b) Prove that if a function $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable at a point $\xi=(x_1,x_2,\ldots,x_n)$ then it is continuous there.

(c) Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\sin x,\forall(x,y)\in\mathbb{R}^2$. Find $f'(1,2)$, where $f'(1,2)$ is the derivative of f at $(1,2)$.

(d) Find the directional derivative of $f(x,y,z)=xy^2-5xz+7y$ at (1, 0, −1) in the direction $\vec{u}=\frac{1}{\sqrt{3}}(1,1,-1)$.

(e) Find the equation of the tangent plane and normal line of the surface $z=x^2+y^2$ at the point $(3,4,25)$.

(f) Evaluate $\iint_R(x^2+y^2)dxdy$, where R is the region bounded by the straight lines $x=0$, $y=1$ and the parabola $y=x^2$.

(g) Evaluate : $\int_0^{\frac{\pi}{2}}dx\ \int_0^{\cos x}x^2dy$

(h) Evaluate : $\int_0^1 dx\ \int_0^y\sqrt{x^2+y^2}\,dy$ by transforming to polar coordinates

(i) Evaluate : $I=\iiint_V\left(xy^2z^3\right)^2dxdydz$ over the rectangular parallelopiped V bounded by the planes $x=\pm a,\ y=\pm b,\ z=\pm c$.

(j) Show that if $\phi(x,y,z)$ is any solution of Laplace's equation then $\vec{\nabla}\phi$ is a vector which is both solenoidal and irrotational.

(k) If $\vec{F}(x,y,z) = xy\hat{\imath} - z\hat{\jmath} + x^2\hat{k}$ and C is the curve $x = t^2,\ y = 2t,\ z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integral $\int_C \vec{F} \times d\vec{r}$.

(l) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{\imath} - 5z\hat{\jmath} + 10x\hat{k}$ along the curve $x = t^2 + 1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$.

(m) Evaluate : $\iint_S \vec{A} \cdot \hat{n}\, ds$ where $\vec{A} = y\hat{\imath} + 2x\hat{\jmath} - z\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

(n) Use Green's Theorem to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(o) If the vectors $\vec{A}$ and $\vec{B}$ be irrotational then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

Group-B

Marks : 20

2. Answer *any four* questions from the following: 5×4=20

(a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0,0) = 0$ and $f(x,y) = \frac{x^2y}{x^4+y^2}$ if $(x,y) \neq (0,0)$. Show that all directional derivatives of f exist at (0, 0), but f is not differentiable at (0, 0).

(b) (i) if $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ then prove that

$$\frac{\partial x}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} + \frac{\partial x}{\partial \phi}\cdot\frac{\partial \phi}{\partial x} = 1.$$

(ii) Let z be a differentiable function of x and y and let $x = r\cos\theta, y = r\sin\theta$. Prove that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

3+2=5

(c) (i) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{xy}{(x^2+y^2)^2}, \text{ for } (x,y) \neq (0,0)$$
$$= 0, \text{ for } (x,y) = (0,0)$$

Show that the repeated integrals of f over $E = \{(x,y): -1 \le x \le 1, -1 \le y \le 1\}$ exist and are equal but $\iint_E f(x,y)dxdy$ does not exist.

(ii) Evaluate : $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx$. 3+2=5

(d) (i) Let $u = f(x,y) = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = -\frac{\sin u \cos 2u}{4\cos^3 u}$.

(ii) Evaluate : $\iint_Q [x+y]dx\, dy$, where $Q = [0,2] \times [0,2]$ and $[x]$ is the greatest integer $\le x$.

3+2=5

(e) Prove that $\vec{F} = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential V such that $\vec{F} = \vec{\nabla}V$. Also find the work done in moving an object in this field from (1, –2, 1) to (3, 1, 4).

1+2+2=5

(f) (i) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region defined by $y = \sqrt{x},\ y = x^2$.

(ii) If S is any closed surface enclosing a volume V and $\vec{A} = ax\hat{\imath} + by\hat{\jmath} + cz\hat{k}$, prove that $\iint_S \vec{A}\cdot\hat{n}\, ds = (a+b+c)V$. 3+2=5

Group-C

Marks : 20

3. Answer *any two* questions from the following: 10×2=20

(a) (i) Let $S \subset \mathbb{R}^n$ be an open ball and let $f: S \to \mathbb{R}$ has partial derivatives $f_{x_1} \dots, f_{x_n}$ at each point of S. If $f_{x_1}, \dots, f_{x_n}$ are continuous at a point $\xi \in S$, then prove that f is differentiable at ξ.

(ii) If $ux = vy = wz = x^2 + y^2 + z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}$, where left hand side of the above relation represents the Jacobian of the functions u, v, w w.r.t. x, y, z.

(iii) Examine for extreme values of $f(x,y) = x^3y^3(12 - 3x - 4y)$ 5+3+2=10

(b) (i) Find the volume of the solid in the first octant bounded by the cylinder $x = 4 - y^2$ and the planes $z = y,\ x = 0,\ z = 0$.

(ii) Show that $\int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}}\, dy = \frac{e-1}{2}$.

(iii) State Schwarz's theorem and show that the function $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$, for $(x,y) \neq (0,0),\ f(0,0) = 0$ does not satisfy its hypothesis. 3+4+3=10

(c) (i) Find the circulation of $\vec{F}$ round the curve C where

$\vec{F} = (2x - y + 4z)\hat{\imath} + (x + y - z^2)\hat{\jmath} + (3x - 2y + 4z^3)\hat{k}$ and C is the circle

$$x^2 + y^2 = 9,\ z = 0$$

(ii) Verify the divergence theorem for $\vec{A} = 2x^2y\hat{\imath} - y^2\hat{\jmath} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

(iii) Verify Stoke's theorem for $\vec{F} = xz\hat{\imath} - y\hat{\jmath} + x^2y\hat{k}$ taken over the surface S of the region bounded by

$x = 0,\ y = 0,\ z = 0,\ 2x + y + 2z = 8$

which is not included in the xz-plane. 3+3+4=10

(d) (i) Find the maximum value of $(xyz)^2$ subject to the condition $x^2 + y^2 + z^2 = 1$ using the method of Lagrange multipliers.

(ii) Evaluate : $\iiint_E \sqrt{a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2}\, dx\, dy\, dz$ where E is the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

5+5=10
