B.A./B.Sc. 6th Semester (Honours) Examination, 2021 (CBCS)
Subject: Mathematics
Course: BMH6CC14
(Ring theory and linear Algebra-II)

Time:3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any six questions: 6×5 = 30

(a) Show that the ring $R = \{m/n : m, n \text{ are integers } and\ n \text{ is odd}\}$ is a principal ideal domain. [5]

(b) Determine $'a'$ such that the polynomial $ax^2 - 4x + 8$ can be expressedas product of irreducible elements in$\mathbb{Q}[x]$. [5]

(c) Is $f(x) = x^4 - 2$ irreducible over the ring $Z[i]$of Gaussian integers? Support your answer. [5]

(d) Let $S = \{(1,0,i), (1,2,1)\}$be a subset of $\mathbb{C}^3$. Compute $S^{\perp}$. [5]

(e) Let $\mathcal{P}_2$be the real inner product space consisting ofall polynomialsover $\mathbb{R}$ of degree ≤ 2 with respect to the inner product,$< f, g >:= \int_0^1 f(t)g(t)dt$. Deduce an orthonormal basis of $\mathcal{P}_2$ with respect to given basis $\{1, t, t^2\}$. [5]

(f) Let V be an inner product space and let W be a finite dimensional subspace of V. If $x \notin$ W, prove that there exists $y \in W^{\perp}$but $< x, y > \neq 0$. [5]

(g) If $f \in (\mathbb{R}^2)^*$is defined by $f(x, y) = 2x + y$ and the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x, y) = (3x + 2y, x)$, then compute $T^t(f)$, where $(\mathbb{R}^2)^*$ is dual of $\mathbb{R}^2$ and T^t, the transpose operator of T. [5]

(h) If W is a subspace of V and $x \notin W$, prove that there exists $f \in W^0$ such that $f(x) \neq 0$, where $W^0 = \{f \in V^* : f(x) = 0,\ \forall x \in W\}$, annihilator of W. [5]

2. Answer any three questions: 10×3 = 30

(a) (i) Show that $\mathbb{Z}[X]/< 1 + X^2 > \cong \mathbb{Z}[i]$, where $< 1 + x^2 >$is the ideal generated by $1 + x^2$. [5]

(ii) Prove that aunitary and upper triangular matrix must be a diagonal matrix. [5]

(b) (i) Let $V = \mathbb{F}^n$ and let $A \in M_{n\times n}(\mathbb{F})$.Prove that $< x, Ay > = < A^*x, y >$ for all $x, y \in V$,where A^* is adjoint of A. [5]

(ii) Factorize $x^p - x$ into irreducible polynomials in $\mathbb{Z}_p[x]$. [5]

(c) (i) Show that $f(x) = x^2 + 8x - 2$ is irreducible over $\mathbb{Q}$. Is it irreducible over $\mathbb{R}$? Support your answer. [5]

(ii) Give an example to show that in a UFD, R,the gcd of two elements a and b of R need not be expressible in the form of $\alpha a + \beta b$, $\alpha, \beta \in R$. [5]

(d) (i) For subspaces $W_1$and W_2 of a vector space V, prove that $W_1 = W_2$ if and only if [5]

$W_1^0 = W_2^0$.

(ii) Suppose that W is a finite dimensional vector space over a field,and $T: V \to W$ is linear. Prove that $N(T^t) = \left(R(T)\right)^0$, where $N(T^t)$, $R(T)$ denotes respectively the kernel of $T^t$and range of T. [3+2]

(e) (i) Let T be a linear operator on an inner product space V, and suppose that $\| T(x) \| = \| x \|$ for all $x \in V$. Prove that T is one-one. [5]

(ii) Let $V = \mathbb{F}^n$ and let $A \in M_{n\times n}(\mathbb{F})$. Suppose that for some $B \in M_{n\times n}(\mathbb{F})$, we have $< x, Ax > = < Bx, y >$ for all $x, y \in V$. Prove that $B = A^*$, where A^* is adjoint of A. [5]