B.A./B.Sc. 2nd Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH2CC03
(Real Analysis)

Time: 3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions $10\times 2 = 20$

(a) Construct a bounded set of real numbers with exactly three limit points. [2]

(b) Is the sequence $\{(-1)^n / n\}$ a Cauchy sequence? Justify your answer. [2]

(c) Give an example of an infinite series $\sum_{n=1}^{\infty} a_n$ such that $(a_1+a_2)+(a_3+a_4)+\dots$ converges but $a_1 + a_2 + a_3 + a_4 + \dots$ diverges. [2]

(d) Determine whether the sequence $\{-2n + \sqrt{4n^2 + n}\}$ is a Cauchy sequence or not. [2]

(e) If $\sum_{n=1}^{\infty} a_n$ converges then prove that $\lim_{n\to\infty} a_n = 0$. Is the converse true? Justify. [2]

(f) Find *sup A* and *inf A*, where $A = \{x \in R: 3x^2 + 8x - 3 < 0\}$. [2]

(g) Show that the set of all even integers is not compact. [2]

(h) If p>0 and t is a real number, then find the limit of the sequence $\{n^t/(1+p)^n\}$. [2]

(i) If y is a positive real number then show that there exists a natural number m such that $0 < 1/2^m < y$. [2]

(j) Find the derived set of the set $S = \{(-1)^n(1+1/n): n \in N\}$ [2]

(k) Show that the sequence $\{(1 - 1/n)\cos(n\pi/2)\}$ is not convergent, but has a convergent subsequence. [2]

(l) State Archimedean property of real numbers and hence show that $\lim_{n\to\infty} \frac{1}{n} = 0$. [2]

(m) Verify that the series $\sum_{n=1}^{\infty} sin\frac{1}{n}$ is not convergent. [2]

(n) Construct an unbounded sequence with exactly one subsequential limit. [2]

(o) If $\{s_n\}$ is a sequence of real numbers and if $s_n \le M$ for all $n \in N$, and if $\lim_{n\to\infty} s_n = L$, then prove that $L \le M$. [2]

2. Answer any four questions $4 \times 5 = 20$

(a) For any two real numbers a, b with $a < b$, prove that there exists a rational number r such that $a < r < b$. [5]

(b) Find the limit superior and limit inferior of the sequence $\{1 + (-1)^n + \frac{1}{2^n}\}$ [3+2]

(c) Prove that every bounded decreasing sequence is convergent [5]

(d) (i) If $\sum_{n=1}^{\infty} a_n$ diverges then prove that $\sum_{n=1}^{\infty} n a_n$ also diverges. [2]

(ii) Let *A* and *B* be two subsets of *R*. If *int A* = *int B* = ϕ and if *A* is closed in *R*, then find *int* (*A*∪*B*). [3]

(e) For any sequence $\{a_n\}$ of positive real numbers, prove that $\lim_{n\to\infty} inf \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \sqrt[n]{a_n}$. [5]

(f) (i) If θ is a rational number, then examine whether the sequence $\{\sin(n\theta\pi)\}$ has a limit. [3]

(ii) If $\sum_{n=1}^{\infty} a_n$ is convergent then test the convergence of the series $\sum_{n=1}^{\infty} \frac{a_n}{\log(n+1)}$. [2]

3. Answer any two questions $2 \times 10 = 20$

(a) (i) Let S be a sequence of real numbers. Show that every subsequence of a subsequence of S is itself a subsequence of S. [5]

(ii) Let a sequence of positive real numbers $\{x_n\}$ converge to x. Prove that the sequence $\{\sqrt{x_n}\}$ converges to $\sqrt{x}$. [5]

(b) (i) Let *S* and *T* be two nonempty bounded subset of *R* such that *S* is a subset of *T*. Prove that $inf\, T \leq inf\, S$. [4]

(ii) Test for convergence of the series $\frac{3}{5}x^2 + \frac{4}{5}x^3 + \frac{15}{17}x^4 + \frac{12}{13}x^5 + \cdots, x > 0$. [6]

(c) (i) If p is a limit point of a subset S of real numbers, then prove that there exists a countably infinite subset of S having p as its only limit point. [5]

(ii) Let *S* be a non-empty subset of real numbers which is bounded below and *T* = {-*x*: *x*∈ *S*}. Prove that the set *T* is bounded above and *Sup T* = - *inf S*. [5]

(d) (i) Let *A* be a subset of *R*. One of the following statements is true and the other is false. Identify the true statement and prove it. Identify the false statement with proper arguments. [3+2]

A. Every interior point of *A* is a limit point of *A*.

B. Every limit point of *A* is an interior point of *A*.

(ii) Examine the convergence of the sequence $\{x_n\}$ where $x_n = \sum_{k=1}^{n} \frac{3k^2+2k}{2^k}$. [5]