B.A./B.Sc. 4th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH4CC10
(Ring Theory & Linear Algebra-1)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions: 10×2 = 20

(a) Does there exists a ring in which the equation $x^2 = x$ has infinitely many solutions? Justify your answer. [2]

(b) Give an example of a finite non-commutative ring? Justify your answer. [2]

(c) Does $(P(X), \cap, \cup)$ form a ring? where $P(X)$ is the power set of the set X. [2]

(d) In the ring $\mathbb{Z}_n$, show that an element 'a' is a unit if and only if $\gcd(a, n) = 1$. [2]

(e) Find out the units in the ring of Gaussian integers $\mathbb{Z}[i]$. [2]

(f) Is the mapping $\varphi: \mathbb{Z}_5 \to \mathbb{Z}_{30}$, defined by $\varphi(x) = 6x, \forall x \in \mathbb{Z}_5$, a ring homomorphism? Justify your answer. [2]

(g) Is the ring $2\mathbb{Z}$ isomorphic to the ring $4\mathbb{Z}$? Justify your answer. [2]

(h) Does there exist an integral domain having exactly 6 elements? Justify your answer. [2]

(i) In $\mathbb{R}^3$, let $v_1 = (2,3,-1)$ and $v_2 = (1,-1,4))$. Examine if $u = (3,7,-6)$ is a linear combination of v_1 and v_2. [2]

(j) Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be a surjective linear map. Show that kernel of T is the singleton. [2]

(k) Does there exist a linear map $T: \mathbb{R}^4 \to \mathbb{R}^4$ whose null space equals with its range? [2]

(l) Is the dimension of $\mathbb{R}$ over the field of rationals $\mathbb{Q}$ finite ? Justify your answer. [2]

(m) Suppose V=$\{f \in C[0,1]: f$ is twice differentiable over$[0,1]$such that $\frac{d^2f}{dt^2} = 0\}$. Then what is the dimesion of the subspace V. [2]

(0) Let V be a vector space over a field F and $v_1, v_2, v_3 \in V$. Suppose that the linear spans of the sets $\{v_1, v_2\}$ and$\{v_1, v_2, v_3\}$ are the same. What can you say about the vectors v_1, v_2, v_3? [2]

2. Answer any four questions: 4×5 = 20

(a) Find all the solution of the equation $x^2 - 5x + 6 = 0$ in the ring$\mathbb{Z}_{14}$. [5]

(b) (i) Is $\mathbb{R}[x]/<x> \cong \mathbb{R}$? Support your answer. [2]

(ii) Is $\mathbb{Z} \times \mathbb{Z}$ an integral domain? Whether the quotient of $\mathbb{Z} \times \mathbb{Z}$ by the ideal $\{0\} \times \mathbb{Z}$ will be an integral domain? Support your answer. [3]

(c) (i) How many non-trivial ideals does the ring $M_{2\times2}(\mathbb{R})$have ? What would be the case if [3] we consider integer entries instead of real entries? Support your answer

(ii) For any two ideals I and J of a ring R, show that I+J will be the smallest ideal in R containing both I and J. [2]

(d) (i) Find a basis of the subspace of $\mathbb{R}^3$ generated by the vectors (1,0,-1), (1,2,1), (0,-3, 2) [2]

(ii) Can you construct a basis of $\mathbb{R}^4$ containing the vectors (2,1,4,3) and (2,1,2,0) ? [3]

(e) (i) Let V, W be two vector spaces over the field $\mathbb{F}$. Let $\mathfrak{B} = \{e_1, e_2, \dots, e_n\}$ be an ordered basis of V and $\mathfrak{B}' = \{f_1, f_2, \dots, f_n\}$ be an ordered basis of W. Can you define a unique linear map T : V → W such that T(e_k)=(f_k), $\forall\, k \epsilon \{1,2, \dots, n\}$. Will it be a linear isomorphism? Justify your answer. [3]

(ii) Suppose W be a subspace of $\mathbb{R}^3$ generated by (1,0,0) and (1,1,0). Can you write down a typical element of the quotient space $\mathbb{R}^3$/W? Also find a basis of $\mathbb{R}^3$/W. [2]

(f) (i) Show that the matrices $\begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}$ and $\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$ are similar over $\mathbb{C}$. [2]

(ii) Let L be a line passing through the origin in $\mathbb{R}^2$. We consider a unit vectorf_1 along L and f_2 is any vector perpendicular to f_1. Find the matrix representation of the linear map T:$\mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that T(f_1)=f_1 and T(f_2)=$-f_2$ with respect to the standard basis of $\mathbb{R}^2$. [3]

3. Answer any two questions: 2×10 = 20

(a) (i) Consider the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients. Let $I = \{f(x) \in \mathbb{Z}[x] : f(0) = 0\}$. Prove that $I = \langle x \rangle$, the principal ideal generated by x. [3]

(ii) State and prove Euler's theorem. Also find out the remainder of 8^{103} when divided by 13. [3+2]

(iii) What is the characteristic of a Boolean ring ? Support your answer. [2]

(b) (i) Find all ideals of ($\mathbb{Z}_{10}, +\,,\ .$) . [2]

(ii) Determine all the ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$. [4]

(iii) State and prove second isomorphism theorem for rings. [4]

(c) (i) Is$\mathbb{Z}[x]$ isomorphic to $\mathbb{R}[x]$? Justify your answer. [2]

(ii) Show that the ideal $< 2, x >$ in $\mathbb{Z}[x]$ will not be a principal ideal. [4]

(iii) Prove that $\mathbb{R}[x] / < x^2 + 1 > \ \cong\ \mathbb{C}$. [4]

(d) (i) Let us consider the ring $R = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b\, \epsilon\, \mathbb{R}\}$. Show that $I = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b\, \epsilon\, \mathbb{R}\}$ is a maximal ideal in R. [3]

(ii) Determine the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that maps the basis vectors $(1,0,0), (0,1,0), (0,0,1)$ of $\mathbb{R}^3$ respectively to the vectors $(0,1,0), (0,0,1), (1,0,0)$. Find $KerT$ and ImT. [3+2+2]