B.A./ B. Sc 4th Semester (Honours) Examination, 2022 (CBCS)

Subject: Mathematics

Course: BMH4CC08

(Riemann Integration and Series of Functions)

Time: 3 Hours Full Marks: 60

The figures in the margin indicate full marks.

Candidates are required to write their answers in their own words as far as practicable.

[Notation and Symbols have their usual meaning]

1. Answer any ten questions: 10×2= 20

(a) Define $f:[0,1]\to\mathbb{Z}$ by $f(x)=[3x]$. Examine whether f is R-integrable. Evaluate $\int_0^1 f(x)\,dx$, if it exists. [2]

(b) Let f be continuous on $[a,b]$ and let $\int_a^b f(x)\,dx=0$. Show that the set $\{x\in[a,b]: f(x)=0\}$ is non-empty. [2]

(c) Let $\phi(t)=\int_0^t te^{\sin t}\,dt, t\in[0,\pi]$. Find $\varphi''(t)$, if exists. [2]

(d) Define $f:[0,\pi]\to\mathbb{R}$ by $f(x)=\sin x$. Find $Sup\,\{|f(a)-f(b)|: a,b\in[0,\pi]\}$. [2]

(e) Find $B(m,1)$ for $m>0$. [2]

(f) Test the convergence of $\int_5^\infty \frac{dx}{x^n}$. [2]

(g) Show that $\int_3^\infty \frac{dx}{x\log x}$ diverges. [2]

(h) Evaluate whether $\int_{\frac{\pi}{2}}^\infty \cos x dx$ is convergent. [2]

(i) Let $\{f_n\}$ be any sequence of functions on $[a,b]$. Is $\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b \lim_{n\to\infty} f_n(x)\,dx$? Give reason. [2]

(j) Examine the pointwise convergence of $\sum_{n=1}^\infty e^{nx}\sin nx$. [2]

(k) Is the series $\sum_{n=1}^\infty \cos nx$ a Fourier series? Justify your answer. [2]

(l) Find the sum of the series $\sum_{n=0}^\infty (2^n+3^n)x^n$, with its range of validity. [2]

(m) If $\sum_{n=0}^\infty a_n x^n$ converges for all real numbers x, then find $\lim_{n\to\infty} Sup\,|a_n|^{\frac{1}{n}}$. [2]

(n) Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. [2]

(o) Find the radius of convergence of $\sum_{n=0}^{\infty} 2^n x^{2n}$. [2]

2. Answer any four questions: 4×5 = 20

(a) (i) Let f be continuous on $[a,b]$ such that $f(x) \geq 0, \forall x \in [a,b]$. If $\int_a^b f(x)dx = 0$, prove that $f(x) = 0, \forall x \in [a,b]$. [3]

(ii) Define f on $[0,1]$ by [2]

$f(x) = 1$, if x is rational in $[0,1]$

=0, otherwise.

Examine whether f is R-integrable.

(b) (i) Let f and g be real valued continuous functions on $[a,b]$ such that [3]

$\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there is a $c \in [a,b]$ such that $f(c) = g(c)$.

(ii) Define $f: [0,1] \to \mathbb{R}$ by [2]

$f(x) = \frac{1}{a^n}$, if $\frac{1}{a^n} < x \leq \frac{1}{a^{n-1}}, n \in \mathbb{N}, a > 1$

=0, if $x = 0$.

Examine whether f is R-integrable.

(c) Let $x \in (-1, 1]$. Show that $\int_0^x \frac{t^m}{t+1}dt \to 0$ as $m \to \infty$. [5]

(d) A function f is defined on $[0,1]$ by $f(0) = 0$ and [5]

$f(x) = (-1)^{n+1}(n+1)$, for $\frac{1}{n+1} < x \leq \frac{1}{n}; n = 1,2,3,....$

Examine the convergence of $\int_0^1 f(x)dx$ and $\int_0^1 |f(x)|dx$.

(e) Show that the sequence of functions $\{f_n\}$ defined by [5]

$f_n(x) = \frac{nx}{1+nx}, x \geq 0$ is not uniformly convergent on $[0,\infty)$.

Examine the uniform convergence of $\{f_n\}$ on $[a,\infty)$ if $a > 0$.

(f) If $f(x) = \{\pi - |x|\}^2$ on $[-\pi,\pi]$, prove that Fourier Series of f is given by [5]

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

3. Answer any two questions: $2\times10 = 20$

(a) (i) State and prove the Second Mean Value Theorem of Integral Calculus in Bonnet's form. [6]

(ii) Evaluate $\lim\limits_{x\to 0}\frac{1}{x^2}\int_0^{x^2} e^{\sqrt{t+1}}dt$. [4]

(b) (i) Show that $\int_0^{\infty} x^{m-1}e^{-x}dx$ is convergent if and only if $m>0$. [5]

(ii) Test the convergence of $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}dx$. [5]

(c) (i) The sequence of functions $\{f_n\}$ is defined on $[0,1]$ by $f_n(x)=x^n-x^{n-1}, x\in[0,1]$. Examine the uniform convergence of $\{f_n\}$ on $[0,1]$. [4]

(ii) For each n, $f_n(x)=1-nx$, for $0\le x\le\frac{1}{n}$ [6]

$=0$, for $\frac{1}{n}<x\le 1$.

Examine the pointwise convergence and the uniform convergence of $\{f_n\}$ on $[0,1]$.

(d) (i) Show that $\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+.....=\frac{1}{1-x}, \forall x, -1<x<1$. [5]

(ii) Expand in a series of sines and cosines of multiples in x of the function given by [5]

$f(x) = x-\pi,$ when $-\pi<x<0$

$=\pi-x,$ when $0<x<\pi$.

What is the sum of the series for $x=\pm\pi$ and $x=0$?