B.A./B.Sc. 4th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH4CC09
(Multivariate Calculus)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions: $10\times 2 = 20$

(a) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist. [2]

(b) Show that the limit exists at the origin but the repeated limits do not, where [2]
$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x}, xy \neq 0 \\ 0,\ xy = 0. \end{cases}$

(c) Show that the function [2]

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y},\ x \neq y \\ 0,\ x = y \end{cases}$$

is discontinuous at $(0,0)$.

(d) Give an example of a function in two variables which is continuous at a point but not differentiable at that point. [2]

(e) Evaluate $\int_0^a \int_0^b xe^{xy}\, dydx$. [2]

(f) Evaluate $\int_0^\pi \int_0^\pi xsinydydx$. [2]

(g) Show that the vector function $\vec{f}(x,y,z) = 3y^4z^2\hat{\imath} + 4x^3z^2\hat{\jmath} - 3x^2y^2\hat{k}$ is solenoidal. [2]

(h) Determine a, b, c so that the vector $\vec{u} = (x+2y+az)\hat{i} + (bx-3y-z)\,\hat{j} + (4x+cy-2z)\widehat{k}$ is irrotational. [2]

(i) Evaluate $\int \vec{F}\cdot\overrightarrow{dr}$,where $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ over the circle $x^2+y^2 = 4,\ z = 0$. [2]

(j) Show that $\iint_S \vec{r}\cdot\hat{n}ds = 4\pi a^3$,where S is the surface of the sphere $x^2+y^2+z^2 = a^2$, $\hat{n}$ is the unit outward normal to S. [2]

(k) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2} = 0$. [2]

(l) Let $z = x^2 + 2xy$. Prove that dzat the point $(1,1)$is given by $dz = 4dx+2dy$. [2]

(m) Let $f(x,y)$ be continuous at an interior point $(a,b)$of the domain of definition of f and let $f(a,b) \neq 0$. Show that there exists a neighbourhood of (a,b)in which $f(x,y)$ retain the same sign as that of $f(a,b)$. [2]

(n) Show that the equation $2xy - log_e(xy) = 2$ determines y uniquely as a function of x near the point $(1,1)$. [2]

(o) Let $f(x,y)=\begin{cases}1, & if\ xy\neq 0\\ 0, & if\ xy=0.\end{cases}$ [2]

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Answer any four questions: $4\times5=20$

(a) Let $u=\frac{x+y}{1-xy},\ v=\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Are they functionally related? If so, find the relationship. [2+3]

(b) If $u=tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $x^2\frac{\partial^2u}{\partial x^2}+2xy\frac{\partial^2u}{\partial x\partial y}+y^2\frac{\partial^2u}{\partial y^2}=sin\,2\,u(1-4\,sin^2\,u)$. [5]

(c) If H is a homogeneous function in x,y,z of degree n and $u=(x^2+y^2+z^2)^{-\frac{1}{2}(n+1)}$,then show that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right)=0$. [5]

(d) Show that $\iiint e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\,dxdydz$taken throughout the region $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$ is $4\pi abc(e-2)$. [5]

(e) Prove that $\vec{\nabla}\times(\vec{\nabla}\times\vec{f})=\vec{\nabla}(\vec{\nabla}\cdot\vec{f})-\nabla^2\vec{f}$. [5]

(f) Evaluate by Stoke's theorem $\oint_\Gamma(sin\,z\,dx-cos\,x\,dy+sin\,y\,dz)$,where $\Gamma$is the boundary of the rectangle $0\leq x\leq\pi,\ 0\leq y\leq 1,\ z=3$. [5]

3. Answer any two questions: $2\times10=20$

(a) (i) State and prove Young's theorem on commutativity of second order partial derivatives. [1+4]

(ii) For the function [5]

$$f(x,y)=\begin{cases}\frac{x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\ ;\end{cases}$$

show that $f_{xy}=f_{yx}$ at $(0,0)$ although neither the conditions of Schwarz's theorem nor the conditions of Young's theorem are satisfied.

(b) (i) Evaluate $\iint_R e^{\frac{y}{x}}\,dxdy$ where R is the triangle bounded by the lines $y=x,y=0$ and $x=1$. [5]

(ii) Evaluate $\iint\frac{dxdy}{(1+x^2+y^2)^2}$over a triangle whose vertices are $(0,0),\ (2,0),\ (1,\sqrt{3})$. [5]

(c) (i) Show that if the vectors $\vec{\alpha},\ \vec{\beta}$ are irrotational, then the vector $\vec{\alpha}\times\vec{\beta}$ is solenoidal. [3]

(ii) Evaluate $\int_C\{xy\hat{\imath}+(x^2+y^2)\hat{\jmath}\}\cdot\overrightarrow{dr}$, where C is the arc of the parabola $y=x^2-4$ from $y=x^2-4$from$(2,0)$to $(4,12)$in the xy-plane and $\vec{r}=(x,y)$. [3]

(iii) Verify Green's theorem for $\oint_C\{(x^2-xy)dx+(y-x^2)dy\}$, where $C=C_1\cup C_2;C_1{:}\,y=x^3,\ C_2{:}\,y=x$. [2+2]

(d) (i) Using divergence theorem show that [4]

$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dv = \iint_S (\phi\nabla\psi - \psi\nabla\phi) \cdot \overrightarrow{ds}$, where ϕ and ψ are continuously differentiable scalar functions and S is the boundary enclosing the region V.

(ii) Show that $\nabla^2(r^n\vec{r}) = n(n+3)r^{n-2}\vec{r}$. [3]

(iii) Give the physical meaning of divergence of a vector function. [3]

(e) (i) State and prove converse of Euler's theorem for homogeneous functions of three variables. [1+3]

(ii) A function $f(x,y)$ becomes $g(u,v)$ where $x = \frac{1}{2}(u+v)$ and $y^2 = uv$. Prove that $\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + 2\frac{x}{y}\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y}\frac{\partial f}{\partial y}\right)$. [3]

(iii) If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that [3]

$$\left(y^2 - zx\right)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$$