B.A/B.Sc. 6th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Course: BMH6CC13
(Metric Spaces and Complex Analysis)

Time: 3 Hours | Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions **10**×2 = 20

(a) Let (X, d) be a metric space. Show that every convergent sequence in X is a Cauchy sequence. [2]

(b) Let C_0 [0,1] be the metric space of all polynomials with real coefficients defined on closed interval [0,1] with distance function [2]

$$d(P, Q) = \underset{0\le t\le 1}{Sup} |P(t) - Q(t)|,$$

where $P, Q \in C_0$ [0,1]. Verify that C_0 [0,1] is not complete.

(c) If $\{x_n\}$ and $\{y_n\}$ are two convergent sequences in a metric space (X, d), show that $\lim\limits_{n\to\infty} d(x_n, y_n) = d\left(\lim\limits_{n\to\infty} x_n, \lim\limits_{n\to\infty} y_n\right)$. [2]

(d) If A and B are two compact sets in $(\mathbb{R}, d)$, prove that $A\times B$ is compact in the Euclidean space $\mathbb{R}^2$. [2]

(e) Examine whether the set $\{(x, y) : x = 0; -2 \le y \le 2\} \cup \left\{(x, y) : 0 < x < 1; y = 2\sin\frac{1}{x}\right\}$ is connected in $\mathbb{R}^2$ with its usual metric. [2]

(f) Let $S = \{(x, y) : x^2 + y^2 = 1\} \cup \{(x, 0) : 1 < x < 2\}$. Examine whether S is connected in $\mathbb{R}^2$ with its usual metric. [2]

(g) Let $X = (0, 1/4)$ be a metric space with the usual metric of $\mathbb{R}$. Let $T: X \to X$ be given by $T(x) = x^2$. Is T a contraction mapping? Is there any fixed point of T in X ? [1+1]

(h) Prove that $f(z) = \begin{cases} \frac{z\,Re(z)}{|z|}, z \neq 0 \\ 0,\ z = 0 \end{cases}$ [2]

is continuous at $z = 0$, where $z \in \mathbb{C}$.

(i) Prove that $f(z) = Im(z),\ z \in \mathbb{C}$, where $z = x + iy$, is nowhere differentiable. [2]

(j) Show that an analytic function over a region with its derivative zero for every point of the region is constant. [2]

(k) If the power series $\sum_{n=0}^{\infty} a_n z^n$ converges to $f(z)$ within its circle of convergence, then show that $a_n = \frac{1}{\underline{|n}} f^n(0)$. [2]

(l) Define e^z and prove that $\frac{d}{dz}(e^z) = e^z$. [1+1]

(m) If C is the circle $|z| = 2$ described in the positive sense and if [1+1]

$$g(z_0) = \oint_C \frac{2z^2 - z + 1}{z - z_0}\, dz,$$

show that $g(1) = 4\pi i$. Find $g(z_0)$ whenever $|z_0| > 2$?

(n) Show that $\int_C f(z)\,dz = 0$, where the contour C is the positively oriented circle $|z| = 1$ and $f(z) = \frac{z^2}{z-4}$. [2]

(o) Prove that $\left|\int_C \frac{e^z}{z+1} dz\right| \le \frac{5\pi e^5}{2}$, where C is the circle $|z| = 5$. [2]

2. Answer any four questions **4**×5 = 20

(a) State and prove Cantor's intersection theorem in a metric space. [1+4]

(b) Let (X, d) be a metric space and A be a nonempty subset of X. Let $f: X \to \mathbb{R}$ be given by $f(x) = d(x, A), x \in X$. Prove that f is uniformly continuous on X. Also show that $f(x) = 0$ if and only if $x \in \overline{A}$. [3+2]

(c) Prove that a sequentially compact metric space is compact. [5]

(d) Find the upper bound for the absolute value of $\oint_C \frac{e^z}{z+1} dz,\ z \in \mathbb{C}$, where C is the circle $|z| = 4$ described in the positive sense. [5]

(e) If $f(z)=u+iv$ is an analytic function of $z=x+iy$ and $u-v=\dfrac{e^y-\cos x+\sin x}{\cosh y-\cos x}$, find $f(z)$ subject to the condition $f\left(\dfrac{\pi}{2}\right)=\dfrac{3-i}{3}$. [5]

(f) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in a Laurent's series, valid for the region $1<|z|<3$. [5]

3. Answer any two questions **2**×10 = 20

(a) (i) State and prove Baire's category theorem in a metric space. [(1+4)+2]

Using Baire's category theorem, show that the set of irrationals with respect to the usual metric of reals is a set of second category.

(ii) Show that every continuous function $f:[-1,1]\to[-1,1]$ has at least one fixed point in $[-1,1]$. [3]

(b) (i) Show that compactness of a metric space implies its sequential compactness. [5]

(ii) Let A be a compact set in a metric space (X,d). Prove that there exist $x,y\in A$ such that $d(x,y)=diam\ A$. [3]

(iii) Prove that every closed subset A of compact metric space (X,d) is compact. [2]

(c) (i) State and prove Cauchy's integral formula. [1+4]

(ii) Evaluate $\oint_C \frac{dz}{z+2}$, where C is the circle $|z|=1$ described in the positive sense. [1+2]

Hence deduce that $\int_0^{\pi}\frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta=0$.

(iii) If $f(z)$ and $\phi(z)$ are analytic in a region R and if they have the same derivative at every point, then show that the functions differ by a constant. [2]

(d) (i) Define an entire function. Prove that every bounded entire function is constant. [1+3]

(ii) Examine the convergence of the series $\sum_{n=0}^{\infty}\frac{z^n}{n!}$, where $z\in\mathbb{C}$. [2]

(iii) Define $sinh\,z$ and $cosh\,z$. Prove that $cosh^2 z-sinh^2 z=1\forall z\in\mathbb{C}$. [2+2]