B.A/B.Sc 6th Semester (Honours) Examination, 2022 (CBCS)
Subject: Mathematics
Paper: BMH6 CC14
(Ring Theory and Linear Algebra-II)

Time: 3 Hours Full Marks: 60

The figures in the margin indicate full marks.
Candidates are required to write their answers in their own words as far as practicable.
[Notation and Symbols have their usual meaning]

1. Answer any ten questions: 10×2= 20

(a) Show that $\mathbb{R}[x]/< x^2+1 > \cong \mathbb{C}$. [2]

(b) Let R be a ring with identity. Prove that R[x] is a PID if R is a field. [2]

(c) Prove that the polynomial $1+ x + x^2+\ldots+ x^n$ is irreducible in $\mathbb{Q}[x]$ if n+1 is a prime number. [2]

(d) If R and S are PIDs, then prove that $R \times S$ is PID. [2]

(e) Prove that $\mathbb{Z}[x]$ is not PID. [2]

(f) Let R be an UFD. Then prove that an element a of R is prime if it is irreducible. [2]

(g) Let R be an integral domain. Then prove that every prime element is irreducible. [2]

(h) Let p be a non-zero element of a commutative ring R with identity. Prove that p is prime if and only if $<p>$ is a prime ideal. [2]

(i) Find two mutually orthogonal vectors each of which is orthogonal to the vector $\alpha = (4,2,3)$ of vector space $\mathbb{R}^3$ over $\mathbb{R}$ with respect to the standard inner product. [2]

(j) Show that every finite dimensional vector space is an inner product space. [2]

(k) If $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ and B={(2,1),(3,1)} is a basis of $\mathbb{R}^2$, find the dual basis of B. [2]

(l) If V is a finite dimensional vector space over a field F and $x_1 \neq x_2$ in V, then prove that $\exists\, f \in V^*$ such that $f(x_1) \neq f(x_2)$. [2]

(m) If V is a vector space over a field F and f is a linear functional from V to F, then show that $f(\theta) = 0$, θ in V and 0 in F. [2]

(n) Prove that in an inner product space the vectors α, β are linearly dependent iff $|< \alpha, \beta >| = \|\alpha\| \|\beta\|$ [2]

(o) Prove that an ideal $<x>$ is maximal ideal in a ring $\mathbb{R}[x]$. [2]

2. Answer any four questions: $4\times5 = 20$

(a) Prove that the ring $\mathbb{Z}[i]$ of Gaussian integers is PID. [5]

(b) Prove that 2 and $1+\sqrt{5}$ are irreducible in $\mathbb{Z}[\sqrt{5}]$. [5]

(c) Prove that any polynomial over $\mathbb{R}$ of degree ≥ 3 is reducible over $\mathbb{R}$. [5]

(d) Give an example of a polynomial which is irreducible over $\mathbb{Z}$ but not irreducible over $\mathbb{Z}_2$. [5]

(e) Let $V= P(\mathbb{R})$, the inner product space of all polynomials with real coefficients over $\mathbb{R}$ with the inner product $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt$. Consider the subspace $P_2(\mathbb{R})$ of all polynomials of degree less than or equal to 2. Using Gram Schmidt orthogonalization process replace the basis $\{1, x, x^2\}$ of $P_2(\mathbb{R})$ by an orthonormal basis of $P_2(\mathbb{R})$. [5]

(f) If f and g are in V^* such that $f(v)=0$ implies $g(v)=0$, prove that $g=\beta f$ for some β in field F. [5]

3. Answer any two questions: $2\times10 = 20$

(a) (i) In $\mathbb{Z}[x]$, prove that $< x >$ is a prime ideal but not a maximal ideal [5]

(ii) Prove that $\mathbb{Z}[x]/<1+x^2>$ is isomorphic with the ring of Gaussian integers $\mathbb{Z}[i]$. [5]

(b) (i) Determine all irreducible polynomials of degree 2 in $\mathbb{Z}_2[x]$. [5]

(ii) Let R be an UFD and P be a nonzero prime ideal of R. Then prove that there exists an irreducible element in P. [5]

(c) Let V be the vector space of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ which have degree less than or equal to 2. Let t_1, t_2, t_3 be three distinct real numbers and L_i: $V\rightarrow \mathbb{R}$ be such that $L_i(p(x))=p(t_i)$, i=1,2,3. Show that $\{L_1,L_2,L_3\}$ is a basis of V^*. Determine a basis for V such that $\{L_1, L_2, L_3\}$ is its dual. [5+5]

(d) (i) Prove that any finite dimensional inner product space has an orthonormal basis. [4]

(ii) Let R be a PID. Then prove that any nonzero proper ideal of ring R can be expressed as finite product of maximal ideals of R. [6]