

B.A./B.Sc. 1st Semester (Honours) Examination, 2022 (CBCS)

Subject : Mathematics

Course : BMH1CC-II

Time: 3 Hours

Full Marks: 60

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.**Notations and symbols have their usual meaning.*

2×10=20

1. Answer any ten questions:

- Find all the values of $(-i)^{\frac{3}{4}}$.
- Solve the equation $x^3 + 6x^2 + 11x + 6 = 0$, the roots are in arithmetic progression.
- If n is a positive integer, then prove that $\left(1 + \frac{1}{n+1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n$.
- Define $f: \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = |x|x, \forall x \in \mathbb{R}$. Examine if f is one-to-one.
- Define an orthogonal matrix. Show that any orthogonal matrix is non-singular.
- Find the rank of the matrix $\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 2 \\ 1 & 4 & 0 \end{pmatrix}$.
- Examine if $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 - y^2 = z^2\}$ is a subspace of \mathbb{R}^3 .
- State Cayley-Hamilton theorem. Verify this for $A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$.
- If A is an $n \times n$ matrix over \mathbb{R} , show that the product of eigenvalues of A is $\det(A)$.
- Show that the real matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has no real eigenvalue.
- Find the remainder when $1! + 2! + 3! + \dots + 50!$ is divided by 15.
- Let a and b be two positive integers such that $\gcd(a, b) = 1$. Find $\gcd(a + b, a - b)$.
- Give an example a binary relation which is reflexive but not symmetric and transitive. Justify your answer.
- Find a basis for the vector space \mathbb{R}^3 that contains the vectors $(1, 2, 0)$ and $(1, 3, 1)$.
- Apply Descartes's rule of signs to examine the nature of the roots of the equation $x^4 + 2x^2 + 3x - 1 = 0$.

2. Answer any four questions:

5×4=20

- State and prove De Moivre's theorem. 1+4
- If x, y, z are positive reals and $x + y + z = 1$, then prove that $8xyz \leq (1-x)(1-y)(1-z) \leq \frac{8}{27}$.
- If P is an orthogonal matrix with $\det(P) = -1$, then show that -1 is an eigenvalue of P .
- Solve the equation $x^3 - 3x - 1 = 0$ by Cardan's method.

- (c) (i) Calculate Sturm's functions and locate the position of the real roots of the equation $x^3 - 3x - 1 = 0$.
- (ii) Let $f: A \rightarrow B$ be an onto mapping and S, T be subsets of B . Then prove that $S \subseteq T \Rightarrow f^{-1}(S) \subseteq f^{-1}(T)$. 3+2
- (f) Show that $W = \{(x, y, z) \in \mathbb{R}^3: x + y - z = 0\}$ is subspace of \mathbb{R}^3 . Find also the dimension of W . 2+3

3. Answer any two questions:

10×2=20

- (a) (i) Find all positive integral solutions of $9x + 7y = 200$.
- (ii) Solve the system of linear equations, if possible:
 $x + 2y + z - 3w = 1$
 $2x + 4y + 3z + w = 3$
 $3x + 6y + 4z - 2w = 5$ 5+5
- (b) (i) Show that the product of all the values of $(1 + \sqrt{3}i)^{\frac{3}{4}}$ is 8.
- (ii) Solve the equation $x^4 + 12x - 5 = 0$ by Ferrari's method. 5+5
- (c) (i) A is a 3×3 real matrix having the eigenvalues 5, 2, 2. The eigenvector of A corresponding to the eigenvalues 5 and 2 are $c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $c \neq 0$ and $c \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $(c, d) \neq (0, 0)$ respectively. Find the matrix A .
- (ii) If every non-zero vector in \mathbb{R}^n is an eigenvector of a real $n \times n$ matrix A , then prove that A is a scalar matrix. 5+5
- (d) (i) A relation R is defined on the set \mathbb{Z} by " $a R b$ iff $ab \geq 0$ " for $a, b \in \mathbb{Z}$. Examine if R is (I) reflexive, (II) Symmetric, (III) Transitive.
- (ii) Prove that a necessary and sufficient condition for a non-homogeneous system $AX = B$ to be consistent is $\text{rank } A = \text{rank } \bar{A}$, where \bar{A} is the augmented matrix of the system. 5+5