B.A./B.Sc. 3rd Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH3CC05

(Theory of Real Functions & Introduction to Metric Spaces)

Time : 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Find $\lim\limits_{x\to 0} e^{x}\,\mathrm{sgn}\left(x+[x]\right)$, where the signum function is defined as sgn $(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

and $[x]$ means the greatest integer less than or equal to x.

(b) Prove, using sequential criteria, that $\lim\limits_{x\to 0} \cos\frac{1}{x^2}$ does not exist.

(c) Give an example of two discontinuous functions $f(x)$ and $g(x)$ such that their product $f(x)g(x)$ is continuous.

(d) Prove that $f(x) = \sin\left(\frac{1}{x}\right)$, $x \in (0,1)$ is not uniformly continuous in $(0,1)$.

(e) Give an example of a real valued function f which is continuous at a point but f' does not exist at that point.

(f) If f' exists and is monotonic on an open interval (a,b), then prove that f' is continuous on (a,b).

(g) Verify Lagrange's mean value theorem for $f(x) = x^3 - 3x + 1$ on $[1,3]$.

(h) If $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a+\theta h)$ and $f^{iv}(x)$ is continuous and non-zero at $x = a$, then show that $\lim\limits_{h\to 0}\theta = \frac{1}{4}$.

(i) Let $f(x) = \begin{cases} 0, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1 \end{cases}$. Does there exist a function F such that $F'(x) = f(x)$ in $[-1,1]$? Justify your answer.

(j) Show that $\cos x + x\sin x > 1$, for $x \in \left(0, \frac{\pi}{2}\right)$.

(k) A function is defined on $[0,1]$ by $f(0) = 1 = f(1)$ and

$$f(x) = 0, \quad \text{if } x \text{ be irrational}$$
$$= \frac{1}{n} \quad \text{if } x = \frac{m}{n}$$

where m, n are positive integers prime to each other.

Examine the continuity of the function $f(x)$ at every rational point in $[0,1]$.

(l) Let d and d^* be metrics on a set X. Examine if $\max\{d, d^*\}$ is a metric on X.

(m) Let (X, d) be a metric space and $A \subset X$. Prove that $\overline{X - A} = X - A^\circ$.

(n) Examine whether the set of rational numbers $\mathbb{Q}$ is closed or not w.r.t. discrete metric.

(o) Let $E = \{(x,y) : 0 < x < 1, 1 < y < 2;\ y \text{ is rational}\}$. Find the set of limit points of E with usual metric in $\mathbb{R}^2$.

2. Answer *any four* questions: 5×4=20

(a) If $f(x)$ is a continuous function on $\mathbb{R}$ such that $f(x+y) = f(x) + f(y)$, find $f(x)$.

(b) Let $f(x)$ and $g(x)$ are both real valued continuous functions defined on $[0,1]$ and let $f(x) = g(x), \forall\, x \in \mathbb{Q} \cap [0,1]$, $\mathbb{Q}$ being the set of rational numbers. Prove that $f(x) = g(x)$, $\forall\, x \in [0,1]$.

(c) Examine the nature of discontinuity of $f(x)$ at $x = 0$, where $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

(d) State and prove Cauchy's mean value theorem. 1+4

(e) (i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

(ii) Show that the function $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ has no Maclaurin's series expansion, although the function has derivative of all orders everywhere. 3+2

(f) Prove that in a metric space arbitrary union of open sets is open. Does this result hold for arbitrary intersections? Justify your answer. 3+2

3. Answer *any two* questions: 10×2=20

(a) (i) Let f be continuous function on a closed and bounded interval $[a, b]$. If $f(a) \neq f(b)$, then prove that f attains every value between $f(a)$ and $f(b)$at least once in the open interval (a, b).

(ii) Give an example to show that intermediate value property does not characterize the continuity of the function.

(iii) Let I be an interval and $f : I \to \mathbb{R}$ be such that f has a relative extremum at an interior point c of I. If $f'(c)$ exists, then prove that $f'(c) = 0$. 3+2+5

(b) (i) Let $D \subset \mathbb{R}$ and a function $f : D \to \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then prove that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

Give an example to show that the above property is not true for continuous function.

(ii) Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$. (3+2)+5

(c) (i) If ρ_1 and ρ_2 are the radii of curvature at the ends of two conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\left(\rho_1^{\frac{2}{3}} + \rho_2^{\frac{2}{3}}\right)(ab)^{\frac{2}{3}} = a^2 + b^2$.

(ii) Find Maclaurin's infinite series expansion of the function $f(x) = \sin x, x \in \mathbb{R}$. 5+5

(d) (i) Let X be a non-empty set and d_1, d_2 be two metrics on X. Prove that $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = \sqrt{(d_1(x,y))^2 + (d_2(x,y))^2}$ is a metric on X.

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