B.A./B.Sc. 3rd Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH3CC06

(Group Theory-I)

Time : 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and Symbols have their usual meaning.

Group-A

1. Answer *any ten* questions: 2×10=20

(a) Let a, b be two elements of a finite group G such that O $(a) = 3$ and $aba^{-1} = b^2$. Find O(b).

(b) Let a be an element of a group G. If $0(a) = n$, then prove that $a, a^2, \ldots, a^n (= e)$ are all distinct.

(c) Show that a group G cannot be the union of its two proper subgroups.

(d) Find the number of generators of the cyclic group $(\mathbb{Z}_{2023},+)$.

(e) Let GL $(2,\mathbb{R})$ be the group of all non-singular 2×2 matrices over $\mathbb{R}$. Prove that $H = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 \neq 0\right\}$ is a subgroup of GL $(2, \mathbb{R})$.

(f) If a and b are elements of a group G with the identity element e such that $ab \neq ba$, then prove that $aba \neq e$.

(g) How many group homomorphisms are there from the group $\mathbb{Z}_6$ to $\mathbb{Z}_5$?

(h) Find all the cosets of the subgroup $4\mathbb{Z}$ in the additive group $\mathbb{Z}$.

(i) Express $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 7 & 5 & 2 & 3 & 1 \end{pmatrix}$ as product of two disjoint cycles. Is it an even permutation?

(j) Find the order of the element $(\bar{2}, \bar{6})$ in the group $\mathbb{Z}_8 \oplus \mathbb{Z}_{12}$.

(k) Is $\mathbb{Z}_2 \oplus \mathbb{Z}_{10}$ isomorphic to $\mathbb{Z}_{20}$? Justify your answer.

(l) Examine if U_{16} is a cyclic group with respect to multiplication modulo 16.

(m) Examine if $\phi : (\mathbb{Z},+) \to (\mathbb{Z},+)$ defined by $\phi(m) = m + 2023$ is a group homomorphism.

(n) Give an example of a non-cyclic group whose all proper subgroups are cyclic.

(o) Let $\phi : (G, 0) \to (G', *)$ be a group homomorphism. Then show that Ker ϕ is a normal subgroup of G.

2. Answer *any four* questions: 5×4=20

(a) Suppose G is a finite cyclic group of order n. Prove that for every divisor d of n, G has only one subgroup of order d.

(b) Let (G, *) be a group. Show that a non-empty subset H of G is a subgroup of G if and only if $a \in H, b \in H \Rightarrow a * b^{-1} \in H$.

(c) Define the centraliser of an element a in a group G and show that it is a subgroup of G.

(d) Let $\phi : G \rightarrow G_1$ be a homomorphism of groups. Then prove that the quotient group G/Ker ϕ is isomorphic to the subgroup Im ϕ of G.

(e) Let (G, 0) be a commutative group of order 8. Prove that the mapping $\phi : G \rightarrow G$ defined by $\phi(x) = x^3 \forall x \in G$ is an isomorphism.

(f) Show that there does not exist an epimorphism from the Klein 4-group to the group to the $\mathbb{Z}_4$.

3. Answer *any two* questions: 10×2=20

(a) (i) If G is a group such that $(ab)^n = a^n b^n$ for three consecutive integers n and for all $a, b \in G$ then prove that G is abelian.

(ii) If H is a subgroup of a cyclic group G, then prove that the factor group G/H is cyclic.

(iii) Let H be a normal subgroup of a group G such that G/H is cyclic. Does it necessarily imply that G is cyclic? Justify your answer. 5+3+2

(b) (i) Show that $D_4 \ncong Q_8$.

(ii) Prove that S_3 has a trivial centre.

(iii) Show that any two finite cyclic groups of the same order are isomorphic. 2+3+5=10

(c) (i) Show that a finite group G has no proper nontrivial subgroups if and only if it is a cyclic group of prime order.

(ii) Let a and b be two elements of a group such that O(a) = 4, O(b) = 2 and $a^3 b = ba$. Show that O(ab) = 2.

(iii) Prove that $\left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\right\}$ is a cyclic subgroup of $GL(2, \mathbb{R})$. 5+3+2=10

(d) (i) Define a dihedral group of degree 4. Let T be the group of all 2×2 invertible matrices over $\mathbb{R}$ under usual matrix multiplication. Let G be the subgroup of T generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that G is a dihedral group of degree 4.

(ii) Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all $g \in G$ is an automorphism if and only if G is abelian.

(iii) Find the number of distinct 3-cycles in S_5 (the set of all permutations on the set {1,2,3,4,5}). 5+3+2=10

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