# B.A./B.Sc. 4th Semester (Honours) Examination, 2023 (CBCS)

## Subject : Mathematics

## Course : BMH4CC09

## (Multivariate Calculus)

**Time: 3 Hours** **Full Marks: 60**

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

*Notation and symbols have their usual meaning.*

### Group-A

(Marks : 20)

1. Answer *any ten* questions: 2×10=20

(a) Show that the function $f(x,y)$ defined by $f(x,y)=\begin{cases}\frac{xy}{\sqrt[3]{x^2+y^2}}, & \text{when } x^2+y^2\neq 0\\ 0, & \text{when } x^2+y^2=0\end{cases}$ is continuous at (0,0).

(b) Show that $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist.

(c) Find $\frac{\partial z}{\partial\theta}$ from the relation $z=\log\sin(x^2y^2-1), x=r\cos\theta, y=r\sin\theta$.

(d) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} y^2dydx$.

(e) Evaluate $\int_1^e\int_1^2\frac{1}{xy}\,dydx$.

(f) Prove that $f(x,y)=|x|+|y|$ is not differentiable at $(0,0)$.

(g) Let $f(x,y)=\frac{x^2y}{x^4+y^2}$. Discuss the existence of the limit of $f(x,y)$ as $(x,y)\to(0,0)$.

(h) Prove that $\lim_{x\to 0}\lim_{y\to 0}\frac{x^2+y^2}{x^2-y^2}\neq\lim_{y\to 0}\lim_{x\to 0}\frac{x^2+y^2}{x^2-y^2}$.

(i) Let $f(x,y)$ be defined as

$$f(x,y)=\begin{cases}1, \text{if } xy\neq 0\\ 0, \text{if } xy=0\,.\end{cases}$$

Prove that $\lim_{(x,y)\to(0,0)}f(x,y)$ does not exist.

(j) If $\vec{a}=(2y+3)\hat{\imath}+xz\hat{\jmath}+(yz-x)\hat{k}$, evaluate $\int_\Gamma\vec{a}.d\vec{r}$, where $\Gamma$ is the curve $x=2t^2$, $y=t, z=t^3$ from $t=0$ to $t=1$.

(k) Find the constants $a,b,c$ so that the vector $\vec{F}=(x+2y+az)\hat{\imath}+(bx-3y-z)\hat{\jmath}+(4x+cy+2z)\hat{k}$ is irrotational.

(l) Use Gauss's divergence theorem to show that $\iint_S \vec{r}.d\vec{s} = 3V$, where $V$ is the volume enclosed by the closed surface $S$ and $\vec{r}$ has its usual meaning.

(m) Show that grad $f$ is a vector perpendicular to the surface $f(x,y,z) = c$, where $c$ is a constant.

(n) If the vectors $\vec{A}$ and $\vec{B}$ are irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

(o) Use Stoke's theorem to prove that $\int_c \vec{r}.d\vec{r} = 0$.

## Group-B

(Marks : 20)

**2.** Answer *any four* questions: 5×4=20

(a) Show that $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$, possesses partial derivatives at (0,0) but is not differentiable at (0,0).

(b) If $\frac{u}{x} = \frac{v}{y} = \frac{w}{z} = (1-r^2)^{-\frac{1}{2}}$ where $r^2 = x^2+y^2+z^2$, then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-\frac{5}{2}}$.

(c) Show that $\iiint e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}dxdydz$ taken throughout the region $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leq 1$ is $4\pi abc(e-2)$.

(d) If $f(0) = 0, f'(x) = \frac{1}{1+x^2}$, prove without using the method of integration that $f(x)+f(y) = f\left(\frac{x+y}{1-xy}\right)$.

(e) Evaluate $\oint_C \vec{F}.d\vec{r}$ by Stoke's theorem where $\vec{F} = y^2\hat{\imath} + x^2\hat{\jmath} - (x+z)\hat{k}$ where $C$ is the boundary of the triangle with vertices $(0,0,0), (1,0,0), (0,1,0)$.

(f) Find the values of the constants $a,b,c$ so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the $z$-axis.

## Group-C

(Marks : 20)

**3.** Answer *any two* questions: 10×2=20

(a) (i) State and prove Young's theorem for commutativity of second order partial derivatives, of a function of two variables.

(ii) Give an example to show that the conditions of the theorem are not necessary. (1+4)+5

(b) (i) State Euler's theorem and its converse for a homogeneous function in $x,y,z$. Use it to prove that if $f(x,y,z)$ is a homogeneous function in $x,y,z$ of degree $n$ having continuous partial derivatives then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are each homogeneous function in $x,y,z$ of degree $n-1$.

(ii) If $H$ is a homogeneous function in $x, y, z$ of degree $n$ and $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}(n+1)}$, then prove that $\frac{\partial}{\partial x}\left(H\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(H\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(H\frac{\partial u}{\partial z}\right) = 0$. 5+5

(c) (i) Prove that for any two vector functions $\vec{f}$ and $\vec{g}$, $div\,(\vec{f} \times \vec{g}) = \vec{g}.curl\,\vec{f} - \vec{f}.curl\vec{g}$.

(ii) Prove that $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla}.\vec{F}) - \vec{\nabla}^2\vec{F}$.

(iii) Give the physical interpretation of divergence of a vector function. 4+4+2

(d) (i) For the function $f$ defined as:

$$f(x,y) = \begin{cases} \frac{1}{y^2}, & \text{if } 0 < x < y < 1 \\ \frac{1}{x^2}, & \text{if } 0 < y < x < 1 \\ 0, & \text{otherwise if } 0 \le x, y \le 1, \end{cases}$$

show that $\int_0^1 dx \int_0^1 f\ dy \neq \int_0^1 dy \int_0^1 f\ dx$. Does the double integral $\iint\limits_R f\,dx\,dy$ exist?

(ii) Find the value of $\iint\limits_E e^{\frac{y}{x}}\ dS$ if the domain $E$ of integration is the triangle bounded by the straight lines $y = x, y = 0$ and $x = 1$.

(iii) Using Green's theorem in the plane, evaluate $\oint_C (2x - y^3)dx - xydy$, where $C$ is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. 4+3+3