B.A./B.Sc. 4th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH4CC08

(Riemann Integration and Series of Functions)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

Group-A

(Marks : 20)

1. Answer *any ten* questions: 2×10=20

(a) Let $f:[1,2]\rightarrow \mathbb{R}$ be continuous on $[1,2]$ and $\int_1^2 f(x)dx=0$. Prove that $\exists\, c\in[1,2]$ such that $f(c)=0$.

(b) Find $\lim\limits_{x\to 0}\frac{1}{x^2}\int_0^x \sin t\, dt$.

(c) If $f:[a,b]\rightarrow\mathbb{R}$ is Riemann integrable on $[a,b]$, then prove that there exists $\mu, m\le\mu\le M$, such that $\int_a^b f(x)dx=\mu\,(b-a)$, where $M=\sup\limits_{a\le x\le b} f(x)$, $m=\inf\limits_{a\le x\le b} f(x)$.

(d) Prove that $\Gamma(n+1)=n\Gamma(n)$.

(e) Let $f:[0,10]\rightarrow\mathbb{R}$ be defined as
$$f(x)=0, \text{when } x\in[0,10]\cap\mathbb{Z}$$
$$=1, \text{when } x\in[0,10]-\mathbb{Z}.$$
Prove that f is Riemann integrable on $[0,10]$ and evaluate $\int_0^{10} f(x)dx$.

(f) Evaluate, if exists $\int_3^7 [x]dx$. ($[x]$ is the highest integer not exceeding x)

(g) Examine the convergence of $\displaystyle\int_0^1 \frac{x^{n-1}}{1+x}\,dx$.

(h) Examine, whether the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on $[0,1]$ is uniform convergent or not, where $f_n(x)=\frac{nx}{n+x}$, $x\in[0,1]$.

(i) Determine the radius of convergence of the power series $+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\cdots$.

(j) A function f is defined on $[0,1]$ as $f(x)=\frac{1}{n}$, if $\frac{1}{n+1}<x\le\frac{1}{n}$, $n=1,2,3,\cdots$
$$=0, \text{ if } x=0.$$
Prove that f is Riemann Integrable on $[0,1]$.

(k) Let $f(x)$ be the sum of the power series $\sum_{n=0}^{\infty}a_nx^n$ on $(-a,a)$ for some $a>0$. If $f(x)=f(-x)$ for all $x\in(-a,a)$, show that $a_n=0$ for all odd n.

(l) Test the convergence of $\int_0^\infty e^{-x^2}dx$.

(m) Examine if $\sum_{n=1}^{\infty} \sin nx$ is a Fourier series or not, give reason in support of your answer, in $[-\pi, \pi]$.

(n) Show that the series $\sum_{n=1}^{\infty} n^{2n}x^n$ converges for no value of x other than 0.

(o) It is given that $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is the Fourier series of the function $f(x) = \frac{1}{2}(\pi - x)$ in $[0, 2\pi]$. What is the value to which the series converges at $x = \frac{\pi}{2}$?

Group-B

(Marks : 20)

2. Answer *any four* questions: 5×4=20

(a) If $f : [a, b] \to \mathbb{R}$ is continuous and $F(x) = \int_a^x f(t)\,dt, x \in [a, b]$, then prove that F is differentiable at any point $c \in [a, b]$ and $F'(c) = f(c)$.

(b) Establish the relation $\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m, n > 0$, where the notations have their usual meaning.

(c) (i) If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge to the same sum function in an interval $(-r, r), r > 0$, then show that $a_n = b_n$, for all n.

(ii) State Dirichlet's condition concerning convergence of Fourier series of a function. 3+2

(d) (i) If $f : [a, b] \to \mathbb{R}$ is continuous, $f(x) \geq 0 \;\forall\, x \in [a, b]$ and $\int_a^b f(x)dx = 0$, then prove that $f(x) = 0 \;\forall\, x \in [a, b]$.

(ii) Show that $\left|\int_a^b \frac{\sin x}{x}\,dx\right| \leq \frac{4}{a}$ for $0 < a < b < \infty$. 3+2

(e) If $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of Riemann integrable functions on $[a, b]$ which converges uniformly to a function f on $[a, b]$, then prove that f is Riemann integrable on $[a, b]$ and $\lim_{n\to\infty} \left(\int_a^b f_n(x)dx\right) = \int_a^b f(x)dx$. 3+2

(f) (i) If the series $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a, b]$, then prove that the series $\sum_{n=1}^{\infty} g(x)f_n(x)$ is uniformly convergent on $[a, b]$, given that g is a bounded function on $[a, b]$.

(ii) Prove that the series $\sum_{n=1}^{\infty} \frac{(n+1)^3}{3^n\, n^5} x^n$ is uniformly convergent on $[-3, 3]$. 3+2

Group-C

(Marks : 20)

3. Answer *any two* questions: 10×2=20

(a) (i) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then prove that $|f|$ is also Riemann integrable on $[a, b]$. Give an example to show that the converse is not true.

(ii) Prove that $\frac{\pi^2}{9} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x}\, dx \le \frac{2\pi^2}{9}$. (4+2)+4

(b) (i) Let $f : [a, b] \to \mathbb{R}$ be a function, $c \in (a, b)$ and f be Riemann integrable on $[a, c]$ and on $[c, b]$. Prove that f is Riemann integrable on $[a, b]$ and $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$.

(ii) State and prove Weierstrass M-test for uniform convergence of a series of functions.

5+(1+4)

(c) (i) Show that the improper integral $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent but not absolutely convergent.

(ii) If a power series $\sum_{n=0}^{\infty} a_n x^n$ has a non-zero radius of convergence, then show that the differentiated series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has also the same radius of convergence.

(iii) Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n!(x+2)^n}{n^n}$. 4+4+2

(d) (i) If a function f is bounded and integrable on $[a, b]$, then prove that $\lim_{n\to\infty} \int_a^b f(x) \cos nx dx = 0$.

(ii) Let $f(x) = \frac{\pi}{4}x, 0 \le x \le \frac{\pi}{2}$

$= \frac{\pi}{4}(\pi - x), \frac{\pi}{2} < x \le \pi$.

Find the Fourier Cosine series of f on $[0, \pi]$. Also deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$.

5+5