B.A./B.Sc. 4th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH4CC10

(Ring Theory & Linear Algebra-1)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) How many idempotent elements does a skew-field have? Support your answer.

(b) Can you give an example of a ring in which an ideal of an ideal of the ring is not an ideal of the given ring? Justify your answer.

(c) Show by an example of a Boolean ring which has infinite number of elements but does not contain unity.

(d) Suppose R, R' be two rings with unity 1, $1'$ respectively and $\varphi : R \to R'$ be a ring homomorphism. Does $\varphi(1) = 1'$ hold good in general? Support your answer.

(e) Does there exist a non-zero ring homomorphism from $\mathbb{Z}$ into $\mathbb{Z}$ except identity? Justify your answer.

(f) Suppose $\varphi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $\varphi(n) = (n, n), \forall n \in \mathbb{Z}$. Is $\varphi(2\mathbb{Z})$ an ideal in $\mathbb{Z} \times \mathbb{Z}$? Support your answer.

(g) Show by an example of a ring without unity will possess a subring with unity.

(h) Does there exist an integral domain having exactly six elements? Justify your answer.

(i) Suppose R be a ring with unity 1 such that $\{0\}$ and R are the only ideals of it. Is R a skew-field? Support your answer.

(j) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map defined by $T(x, y) = (e^x, e^y)$. Is it one-one? Support your answer.

(k) Does there exist a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that *range* $(T) = \mathcal{L}S\{(1, \pi)\}$? Justify your answer.

(l) Write down the matrix representation of the linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y - z, x + z)$ with respect to the standard bases.

(m) Suppose $V = \{f \in C[0,1] : f$ is twice differentiable over [0, 1] such that $\frac{d^2 f}{dt^2} + 1 = 0\}$. What is the dimension of the subspace V?

(n) Examine if the set $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ linearly independent in the vector space $\mathbb{F}^3$, where $\mathbb{F}$ is a field with characteristic 2? Justify your answer.

(o) What are the proper subspaces of $\mathbb{R}^3$? Support your answer.

2. Answer *any four* questions: 5×4=20

(a) (i) Let R be a ring with unity 1 such that the set of all non-units in R forms a subgroup of the additive group $(R, +)$. Prove or disprove either $Char\,(R) = 0$ or it is a power of a prime.

(ii) Examine if the sum of two zero-divisors in a ring is again a zero-divisor. 3+2

(b) (i) Is the ideal $< x >$ maximal in the polynomial ring $\mathbb{Z}[x]$? What will happen if we consider the same in the polynomial ring $\mathbb{R}[x]$? Support your answer.

(ii) Is the ideal $< x^2 + 1 >$ prime in the ring $\mathbb{Z}[x]$? Is it maximal? Justify your answer. 2+3

(c) (i) Determine all the maximal ideals of the Euclidean Plane $\mathbb{R}^2$.

(ii) Find a basis of the subspace of $\mathbb{R}^3$ generated by the vectors (1,0,–1), (1,2,1) and (0,–3,2). 3+2

(d) (i) Examine if $M_{\frac{1}{2}} = \left\{f \in C[0,1] : f\left(\frac{1}{2}\right) = 0\right\}$ is a maximal ideal in the ring of continuous functions $C[0,1]$.

(ii) Let z be a complex number such that $Im(z) \neq 0$. Does the set $\{z, z^2\}$ form a basis of the field $\mathbb{C}$ over the real field $\mathbb{R}$? 3+2

(e) (i) Examine if the map $T : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n+1}(\mathbb{R})$ defined by $T(p(x)) = xp(x), p \in \mathcal{P}_n, x \in \mathbb{R}$, is linear.

(ii) Let L be a line passing through the origin in $\mathbb{R}^2$. Find out the nullity and rank of the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ which satisfies $T(0) = 0$ and T maps each point onto its reflection with respect to the given line L. 2+3

(f) (i) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map which rotates every point through the same angle ϕ about the origin O. Is it linear? If yes, what will be the nullity and rank of T?

(ii) Is the map $T(z) = \bar{z} =$ the complex conjugate of z, linear over the real vector space $\mathbb{C}$? Is it linear if we consider the field $\mathbb{C}$ over itself? Justify your answer. 3+2

3. Answer *any two* questions: 10×2=20

(a) (i) Find the characteristic of each of the following rings:

(1) $5\mathbb{Z}$, (2) $(\mathcal{P}(X), +, \cap)$, for a given non-empty set X.

(ii) Determine all the subrings of $(\mathbb{Z}_7, +, \ .)$.

(iii) What is the characteristic of a Boolean ring? Support your answer. 4+4+2

(b) (i) Find all ideals of $(\mathbb{Z}_{10}, +, \ .)$.

(ii) For a ring R, let us consider the radical *rad* (R) which is defined to be the set of all nilpotent elements of the ring R. Does it form an ideal in R? Also compute *rad* $(\mathbb{Z})$.

(iii) Let $\mathcal{C}$ be the linear space of all real convergent sequences. Define a map $T : \mathcal{C} \to \mathcal{C}$ by the rule $T(x) = \{y_n\}$, where $y_n = \lim x_n - x_n$, for all $x = \{x_n\} \in \mathcal{C}$. Show that T is linear. Also find its null space and range. 2+4+4

(c) (i) If every ideal in a commutative ring with unity is prime, then prove that it is a field.

(ii) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $T(1,0) = (1,0,1), T(0,1) = (1,1,1)$. Determine the rank and nullity of T. Also find its matrix representation relative to standard bases.

(iii) Prove that there exists infinitely many linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,-1,1) = (1,2)$ and $T(-1,1,2) = (1,0)$. 3+4+3

(d) (i) If V and W are two n-dimensional vector spaces over a *field* $\mathbb{F}$, then show that a linear map $T : V \to W$ is non-singular if and only if *rank* $(T) = n$. Does it hold if we drop finite dimensionality of the vector spaces V and W? Justify your answer.

(ii) Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be a map defined by $T(f(x)) = 2f'(x) + \int_0^x 3f(u)du$. Show that T is a linear map. Also find *Ker* (T) and *range* (T). Is it invertible? Justify your answer.

(iii) Let $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be defined by $T(A) = A^t, \forall A \in M_{2\times2}(\mathbb{R})$. Choose whether it is linear or not. If yes, find the matrix representation of T with respect to the standard basis of $M_{2\times2}(\mathbb{R})$. 4+3+3