B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6CC-XIII

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x,0) : 1 < x < 2\}$. Examine whether S is connected in $\mathbb{R}^2$ with its usual metric.

(b) Examine whether the set $\{(x,y) \in \mathbb{R}^2 : 0 < x < 1;\ x \text{ is rational};\ x = y\}$ is complete in $\mathbb{R}^2$ with its usual metric.

(c) Using the definition of compactness, prove that the open interval (1, 2) is not compact in $\mathbb{R}$.

(d) If f is a real valued function on $X = \left[0, \frac{1}{3}\right]$ with usual metric, defined by $f(x) = x^2$, then show that f is a contraction mapping on X.

(e) Give an example to show that the continuous image of a Cauchy sequence need not be a Cauchy sequence.

(f) Prove that a contraction mapping $T: (X,d) \to (X,d)$ is uniformly continuous.

(g) Let $f: X \to \mathbb{R}$ be a non-constant continuous function, where (X,d) is connected. Prove that $f(X)$ is uncountable.

(h) Evaluate $\int_C \frac{dz}{z}$, where C is the unit circle $|z| = 1$.

(i) Define $\sin z$ and prove that $\frac{d}{dz}(\sin z) = \cos z$. 1+1

(j) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^{-n}}{1+in^2} z^n$.

(k) Let f be analytic in a connected domain $D \subset \mathbb{C}$ and $f'(z) = 0\ \forall z \in D$. Prove that f is constant on D.

(l) Prove that $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$.

(m) Prove that $f(z) = \begin{cases} \frac{zRe(z)}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ is continuous at $z = 0$.

(n) Show that $\int_C f(z)dz = 0$, where C is the unit circle $|z| = 1$ in the positive direction and $f(z) = \frac{z^2}{z-6}$.

(o) Find the maximum modulus of $f(z) = 2z + 5i$ on the closed region: $|z| \leq 2$.

2. Answer *any four* of the following: 5×4=20

(a) Let $f: (X, d_1) \to (Y, d_2)$ be a function, then show that f is continuous if and only if $f^{-1}(G)$ is open in (X, d_1) whenever G is open in (Y, d_2).

(b) Show that continuous image of a connected subset in domain space is connected in range space.

(c) Let A and B be two nonempty subsets of a metric space (X, d), where B is compact. Prove that $d(A, B) = 0$ if and only if $\bar{A} \cap B \neq \phi$.

(d) Let $f: G \to \mathbb{C}$ be an analytic function on region G such that $|f(z)|$ is constant on G. Show that f is constant on G.

(e) If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.

(f) Evaluate $\int_C f(z)dz$ where C is the positively oriented circle $C: |z - i| = 2$ and $f(z)$ are the following: 2+3

(i) $f(z) = \frac{1}{z^2+4}$

(ii) $f(z) = \frac{1}{(z^2+4)^2}$

3. Answer *any two* questions: 10×2=20

(a) (i) Define a Lebesgue number. Prove that in a sequentially compact metric space, every open cover has a Lebesgue number.

(ii) Let A be a compact set of diameter $\delta(A)$. Prove that there exist a pair of points $x, y \in A$ such that $\delta(A) = d(x, y)$. Is compactness of A necessary to hold the above result? Justify your answer. (1+4)+(4+1)

(b) (i) Show that the unit sphere $S = \{x = \{x_n\} \epsilon l_2 : \sum_{n=1}^{\infty} x_n^2 \leq 1\}$ is not compact.

(ii) Show that the function $f(z) = e^{-z^{-4}} (z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at the point. 5+5

(c) (i) Let $f(z)$ be an entire function. If real part of $f(z)$ is bounded, prove that $f(z)$ is constant.

(ii) State and prove 'Banach Fixed Point Theorem'. 4+6

(d) (i) If $p(z)$ be a non-constant polynomial of degree n, then show that there is a complex number α satisfying $p(\alpha) = 0$.

(ii) Evaluate

$$\oint_C \frac{dz}{z(z + \Pi i)}$$

where C is equal to $\{z: |z + 3i| = 1\}$. 5+5
