B.A./B.Sc. 6th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH6DSE42

(Differential Geometry)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Define a space curve. Give an example of it.

(b) When is a curve in $\mathbb{R}^n$ said to be ~~unit~~ speed? Give an example of a unit speed curve.

(c) Define arc length of a plane curve. Find the arc length of the curve $\gamma(t) = (e^t \cos t, e^t \sin t)$ at $\gamma(0) = (1,0)$.

(d) Deduce the curvature of the curve $\gamma(t) = (\cos^3 t, \sin^3 t)$.

(e) Define signed curvature of a plane curve.

(f) Define a surface immersed in $\mathbb{E}^3$.

(g) Show that the surface of a sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is a smooth surface.

(h) Deduce the first fundamental form of a plane.

(i) When is a curve on a surface said to be geodesic? Prove that any geodesic has constant speed.

(j) State second fundamental form of a surface $\sigma(u,v)$.

(k) When is a point on a surface said to be umbilic?

(l) State Meusnier's theorem.

(m) Give an example of a surface of positive curvature.

(n) Define mean curvature of a surface.

(o) Give an example of a surface whose Gaussian curvature and mean curvature are different.

2. Answer *any four* questions: 5×4=20

(a) If γ is a unit speed curve of $\mathbb{R}^3$ with constant curvature and zero torsion, then prove that γ is a part of a circle.

(b) Deduce the torsion of a curve $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

(c) Deduce the first fundamental form of a surface $\sigma(u,v) = (u, v, u^2 + v^2)$.

(d) Prove that any tangent developable surface is isometric to a plane.

(e) Prove that a curve on a surface is a geodesic if and only if its geodesic curvature is zero everywhere.

(f) Prove that the Gaussian curvature of a ruled surface is negative or zero.

3. Answer *any two* questions: 10×2=20

(a) Let $\gamma(t)$ be a unit speed curve with $k(t) > 0$ and $\tau(t) \neq 0$ for all t. Show that γ lies on the surface of a sphere of radius r if and only if

$$\frac{\tau}{k} = \frac{d}{ds}\left(\frac{\dot{k}}{\tau k^2}\right),$$

where $r^2 = \rho^2 + (\dot{\rho}\sigma)^2, \rho = \frac{1}{k}, \sigma = \frac{1}{\tau}$ and dot $(\cdot)$ denotes the differentiation. 5+5

(b) Obtain a necessary and sufficient condition for a space curve to be a helix. 5+5

(c) State and prove Euler's theorem on a surface. 2+8

(d) Deduce the Gaussian curvature of the helicoid $\sigma(u, v) = (v \cos u, v \sin u, \lambda u)$ and catenoid $\sigma(u, v) = (\cos hu \cos v, \cos hu \sin v, u)$. 5+5
