# B.A./B.Sc. 3rd Semester (Honours) Examination, 2019 (CBCS)

## Subject : Mathematics

## Paper : BMH3 CC-05

**Time: 3 Hours** **Full Marks: 60**

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

*(Notations and Symbols have their usual meaning.)*

### Group A

**1.** Answer *any ten* questions: 2×10=20

(a) Prove that $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist.

(b) Let $I$ be an interval and $f: I \to \mathbb{R}$ be continuous on $I$. Then prove that $f(I)$ is an interval.

(c) Prove that $f(x) = \sin\frac{1}{x}, x \in (0,1)$ is not uniformly continuous on (0, 1).

(d) Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable at $c \in \mathbb{R}$ and $f(c) = 0$. Let $g(x) = |f(x)|,\ x \in \mathbb{R}$. Show that $g$ is differentiable at $c$ if and only if $f'(c) = 0$.

(e) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 sin\frac{1}{x} \text{ for } x \neq 0$$
$$= 0 \quad , \text{for } x = 0$$

Show that $f$ is differentiable on $\mathbb{R}$ but $f'$ is not continuous on $\mathbb{R}$.

(f) If $p(x)$ is a polynomial of degree > 1 and $k \in \mathbb{R}$, prove that between any two real roots of $p(x)$ there is a real root of $p'(x) + kp(x) = 0$.

(g) If $x, y, x'$ and $y'$ are elements in a metric space $(X, d)$, show that $|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y')$.

(h) Is $L'$ Hospital's rule applicable to find $\lim_{x\to 0} \frac{\sin x}{x}$? Justify.

(i) Prove that the equation $x = \cos x$ has a root in $\left(0, \frac{\pi}{2}\right)$.

(j) Give example with proper justification, of two discontinuous functions whose product is a continuous function.

(k) Show that $f(x) = [x]$ in $[0, 1]$ is not the derivative of any function, $[x]$ being the largest integer not larger than $x$.

(l) Give the geometrical significance of Cauchy's Mean Value Theorem.

(m) Prove or disprove : Every subset in a discrete metric space in closed.

(n) Let $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $d(x, y) = |x^3 - y^3|, \forall\, x, y \in \mathbb{R}$ . Is '$d$' a metric on $\mathbb{R}$? Support your answer.

(o) Let $S = \{(x, y): 0 < x < \frac{2\pi}{3}; y = \sin x\}$. Find the diameter of $S$ on $\mathbb{R}^2$.

## Group B

**2.** Answer *any four* questions: 5×4=20

(a) (i) A function $f: [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = x, x \text{ is rational in } [0, 1],$$
$$= 1 - x, x \text{ is irrational in } [0, 1].$$

Show that $f$ is continuous at $\frac{1}{2}$ and discontinuous at every other point in [0, 1].

(ii) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that $f^{-1}(G)$ is open in $\mathbb{R}$ for every open subset $G$ of $\mathbb{R}$. (1+2)+2=5

(b) (i) Let $f: \mathbb{R} \to \mathbb{R}$ be a function. If $f''(x) > 0$ for all $x \in \mathbb{R}$, then show that $f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]$.

(ii) If $f(x)$ is defined in [0, 1] such that $f(0) = \frac{1}{2}$. Also if $f$ is continuous and takes up only rational values, then show that $f(x) = \frac{1}{2}$ for all $x \in [0, 1]$. 3+2=5

(c) State and prove Taylor's Theorem with Cauchy's form of remainder. 1+4=5

(d) Prove that every continuous function on a closed and bounded interval is uniformly continuous. 5

(e) State and prove Darboux theorem concerning derivative. 1+4=5

(f) Let $(X, d)$ be a metric space. Define a closed set in $(X, d)$. Prove that arbitrary intersection of closed sets in $(X, d)$ is a closed set. Is arbitrary union of closed sets in a metric space closed? Support your answer. 1+2+2=5

## Group C

**3.** Answer *any two* questions: 10×2=20

(a) (i) Let $c[a, b]$ be the set of all real valued continuous functions on $[a, b]$ and $d$ be a metric on $c[a, b]$ defined by $d(f, g) = \sup\limits_{a \leq x \leq b} |f(x) - g(x)|$ for all $f, g \in c[a, b]$. Show that the set $\left\{f \in c[a, b]: \inf\limits_{a \leq x \leq b} f(x) > 0\right\}$ is an open set in $c[a, b]$ with respect to the metric $d$.

(ii) Use Mean Value Theorem to prove $0 < \frac{1}{x}\log\left(\frac{e^x-1}{x}\right) < 1$, for $x > 0$.

(iii) Show that the function defined on$(-1, 2)$ by $f(x) = \begin{cases} x & , \text{ when } -1 < x < 1 \\ x-2 & , \text{ when } 1 \le x < 2 \end{cases}$

has minimum at $x = 1$. 4+3+3=10

(b) (i) If $\rho_1$ and $\rho_2$ be the radii of curvature at the extremities of any chord of the cardiode $r = a(1+\cos\theta)$, which passes through the pole, then prove that $\rho_1^2 + \rho_2^2 = \frac{16}{9}a^2$.

(ii) Find the equation of the circle of curvature of $2xy + x + y = 4$ at the point (1, 1).

5+5=10

(c) (i) Show that the metric space $l^p, 1 \le p < \infty$ is separable.

(ii) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ and $f$ be continuous at a point of $\mathbb{R}$ ; then prove that $f$ is uniformly continuous on $\mathbb{R}$.

(iii) If $f(x) = \sin x$ prove that $\lim\limits_{h \to 0} \theta = \frac{1}{\sqrt{3}}$, where $\theta$ is given by

$f(h) = f(0) + hf'(\theta h), 0 < \theta < 1$. 4+3+3=10

(d) (i) Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ be a function. Let $c \in D \cap D'$. Then prove that $f$ is continuous at $c$ if and only if for every sequence $\{x_n\}$ in $D$ converging to $c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

(ii) Let $I$ be an interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ be a function. Let $f''$ exist in $I$. Then prove that $f$ is convex in $I$ if and only if $f''$ is non-negative in $I$. 5+5=10