B.A./B.Sc. 3rd Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

(Group Theory-I)

Paper : BMH3 CC-06

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

(Notations and Symbols have their usual meaning.)

Group A

1. Answer *any ten* questions: 2×10=20

(a) Give examples of two non-Abelian groups of order 8.

(b) Show that $\mathbb{Z}_4 = \langle 3 \rangle$.

(c) Give an example of infinite group of which every element is of finite order.

(d) If G is a group and H is a subgroup of index 2, prove that H is a normal subgroup of G.

(e) Let a be an element of order 3 in a group G with indentity e. If $a^k = e$, where are the possible values of k?

(f) Find the order of each element of the group $\mathbb{Q}$ of rational numbers under addition.

(g) What is the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{pmatrix}?$$

(h) Express the permutation (1 2 3 4 5) as a product of 2-cycles in two different ways.

(i) What is the order of a permutation which can be expressed as the product of two disjoint cycles of lengths 4 and 6?

(j) Suppose G is a cyclic group of order 20. How many subgroups does G have?

(k) What is the order of any non-identity element of $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$?

(l) What is the order of $5 + \langle 6 \rangle$ in the factor group $\mathbb{Z}_{18}/\langle 6 \rangle$?

(m) Find all the generators of $\mathbb{Z}_{10}$.

(n) Let a, b be two elements of a finite group G such that $\circ\,(a) = 5, b \neq e$ and $aba^{-1} = b^2$. Find $\circ\,(b)$, where $\circ\,(c)$ is the order of $c \in G$.

(o) Find $\alpha \circ \beta \circ \alpha^{-1}$ where $\alpha = (1\ 3\ 5\ 7)$ and $\beta = (2\ 4\ 8) \circ (1\ 3\ 6) \in S_7$.

Group B

2. Answer *any four* questions: 5×4=20

(a) Prove that the order of the subgroup H of a finite group G divides the order of the group G. Does the converse hold? Support your answer. 3+2=5

(b) Define the centre $Z(G)$ of a group G. Prove that $Z(G)$ is normal subgroup of the group G. 1+4=5

(c) Show that the set $S = \left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right\}$ is a group under matrix multiplication. Is it Abelian? Justify your answer. 3+2=5

(d) Show that A_5 has no normal subgroup. 5

(e) Let G_1 and G_2 be groups and $\varphi : G_1 \to G_2$ be an isomorphism. Prove that for every $a \in G_1$, the order of a is the same as the order of $\varphi(a)$. Use this fact to prove that $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. 3+2=5

(f) Prove that the mapping $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$, defined by $\varphi\left((a, b)\right) = a - b \;\forall\, (a, b) \in \mathbb{Z} \oplus \mathbb{Z}$, is a homomorphism. What is the Kernel of ϕ? 3+2=5

Group C

3. Answer *any two* questions: 10×2=20

(a) (i) Find all homomorphism from $(Z_8, +)$ to $(Z_6, +)$.

(ii) Let H and K be two subgroups of a group G. Suppose K is normal on G. Prove that $\dfrac{H}{H \cap K} \simeq \dfrac{HK}{K}$.

(iii) Is $\mathbb{R}$ under addition a cyclic group? Support your answer. 3+5+2=10

(b) (i) Let $G = \langle\, a \,\rangle$ be a cyclic group of order n. Prove that $G = \langle\, a^k \,\rangle$ iff $gcd\,(k, n) = 1$.

(ii) List all the subgroups of $\mathbb{Z}_{30}$.

(iii) Let a be an element of a group and the order of a be 15. Compute the orders of a^9 and a^4 . 5+3+2=10

(c) (i) Prove that the intersection of two normal subgroups of a group G is a normal subgroup of G.

(ii) If M and N are normal subgroups of a group G then prove that MN is a normal subgroup of G.

(iii) If G is a finite group and N is a normal subgroup of G, prove that $\circ\left(\frac{G}{N}\right) = \frac{\circ(G)}{\circ(N)}$.

($\circ\,(A)$ denote the order of A). 4+4+2=10

(d) (i) Let G be a group and $Z(G)$ be the centre of G. If the quotient group $G/Z(G)$ is cyclic then prove that G is Abelian. Hence prove that if G is a non-Abelian group of order p^3 (p being a prime) then the order of $Z(G)$ is 1 or p.

(ii) Find the centre of the dihedral group D_4.

(iii) Does the group of integers under addition have a finite subgroup? Support your answer. (4+2)+2+2=10