B.A./B.Sc. 5th Semester (Honours) Examination, 2019 (CBCS)

Subject : Mathematics

Paper : BMH5DSE13

(Point Set Topology)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions from the following: 2×10=20

(a) Show that the set of real numbers $\mathbb{R}$ is uncountable.

(b) Show that the set $\mathbb{Z}$, the set of all integers, is equipotent with $\mathbb{N}$, the set of natural numbers.

(c) Show that the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ is closed in $(\mathbb{R}, \tau_1)$ with upper limit topology τ.

(d) Is the set $[0, 4) \cup [4,10]$ connected in the real number space with usual topology? Support the answer.

(e) Give an example to show that continuous image of a locally connected space need not be locally connected.

(f) Give an example of a locally compact space which is not compact.

(g) Let X be an infinite set and $\tau = \{A \subset X : X - A \text{ is finite}\} \cup \{\varphi\}$. Examine whether τ is a topology on X.

(h) *Let* (x, τ) and (y, τ^1) be two topological spaces. Then prove that $f: (x, \tau) \to (y, \tau^1)$ is continuous if $f^{-1}(G) \in \tau \; \forall \; G \in S^1$, where S^1 is a sub base of τ^1.

(i) *Let* $(\mathbb{R}, \tau)$ be a real number space with usual topology and $f: (x, \tau^1) \to (\mathbb{R}, \tau)$ is continuous. Show that the set $\{x \in X : |f(x)| < 1\}$ is open in the topological space (x, τ^1).

(j) *Let* (X, τ) be a connected topological space. Prove that there is no continuous surjection $f: X \to \{0, 1\}$ where $\{0, 1\}$ denotes the discrete space of two elements.

(k) Prove that each co-finite space is compact.

(l) *Let* (X, τ) be a topological space and $A \subset X$. Prove that $(X - A) = X - int\, A$.

(m) Give examples of two disjoint dense subsets in the space $\mathbb{R}$ of reals with respect to the usual topology.

(n) Define a locally compact space. Is local compactness a hereditary property? Justify.

(o) *Let* $f: X \to Y, g: Y \to Z$ be continuous functions. Show that $g \circ f: X \to Z$ is continuous.

2. Answer *any four* questions: 5×4=20

(a) (i) If $\cup$ be a set and $P(\cup)$ be the power set of $\cup$, then show that $\bar{\bar{\cup}} < \overline{\overline{P(\cup)}}$ where $\bar{\bar{A}}$ denotes the cardinal number of A.

(ii) $Let\ (X,\tau)$ be a topological space. Show that a subfamily B of τ forms an open base of τ if for any open set G and for any point $p \in G, \ni V \in B$ such that $p \in V \subset G$. 3+2=5

(b) State and prove Schröder-Bernstein theorem. 1+4=5

(c) (i) Let X, Y be two topological spaces and $f: X \to Y$ be continuous. $Let\ A$ be a connected subset in X. Prove that $f(A)$ is connected in Y.

(ii) Prove that in a topological space closure of a connected set is connected. 3+2=5

(d) Define component in a topological (X,τ). Prove that every component in (X,τ) is a closed set. Is component of a space always an open set in the space? Justify the answer. 1+2+2=5

(e) Define a path connected space. Prove that every path connected space is connected. 1+4=5

(f) Define an ϵ-net in a metric space. When is a metric space said to be totally bounded? Prove that every totally bounded metric space is bounded. 1+1+3=5

3. Answer *any two* questions from the following: 10×2=20

(a) (i) $Let\ u$ be the cardinal number of a set U. Prove that the power set $P(U)$ has cardinal number 2^u.

(ii) $Let\ \alpha, \beta, \gamma$ be three ordinal numbers. Is $\alpha + \beta = \beta + \alpha$? Support your answer. Prove that $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. 5+(3+2)=10

(b) (i) Prove that a subset of $\mathbb{R}$ is connected if and only if it is an interval.

(ii) Prove that a closed subset of a compact space is compact. 5+5=10

(c) (i) State Baire category theorem.

(ii) Can the plane be expressed as a countable union of straight lines? Support your answer.

(iii) Is a metric space containing finitely many elements of first category? Support your answer.

(iv) Give an example of a metric space of first category. 1+4+4+1=10

(d) (i) $Let\ (X,\tau)$ be the topological product of the family of topological spaces $\{(X_i,\tau_i): i = 1, 2, \dots n\}$. Show that the projection mapping $p_i: (X,\tau) \to (X_i,\tau_i)$ is open mapping and continuous.

(ii) $Let (A,\tau_A)$ be a subspace of a topological space (X,τ). Then show that a subset F is closed in (A,τ_A) if and only if $F = A \cap K$ where K is a closed set in (X,τ).

(iii) Give an example with justification of a locally connected space which is not connected. (2+2)+4+2=10

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