LAPLACE TRANSFORM AND ITS APPLICATION TO SOLVE INITIAL AND BOUNDARY VALUE PROBLEMS

Project Submitted By - Satriya Bakshi

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Roll- 200313700008

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Government General Degree College At Kalna – 1

Under Guidance Of Dr. Dibakar Mondal,

Assistant Professor of Mathematics, GGDCK-1,

University Of Burdwan

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Satriya Bakshi

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INTRODUCTION

In Mathematics, the Laplace Transform is an integral transform named after its inventor French mathematician Pierre Simon Laplace (1749-1827), and systematically developed by the British Physicist Oliver Heaviside (1850-1925), to simplify the solution of many differential equations that describe physical processes .The technique of integral transform is a powerful and indispensable tool for modern applied mathematics and for theoretical physicist for successful investigation of boundary value problems arising in mathematical physics. Laplace or Fourier transforms are generally used to solve a boundary value problem where the governing partial differential equation is linear such as the Laplace's equation or the modified Helmholtz equation in the Rectangular Cartesian Co-ordinate system. Most of the integral transforms and their inversion formula are available in standard books on integral transform.

- WHAT WE WANT TO STUDY IN THIS PROJECT~

In this project work we will describe the application of Laplace transformation in solving differential equation theoretically step by step manner. First we define it,then derive it conditions under which it exists. After that we obtain Laplace transformation and inverse of some elementary function. Then we discuss Laplace transform of derivative of a function which is very important in this context. This will be followed by procedure of solving differential equation. Next we validate this procedure by discussion of a particular differential equation.

➤ Piecewise Continuous Function ~ A function f(t) is said to be a Piecewise continuous function on a closed interval $a \leq t \leq b$ if the interval can be subdivided into finite number of intervals in each of which f(t) is continuous and has finite left and right hand limits.

- Examples: Consider the function f defined by

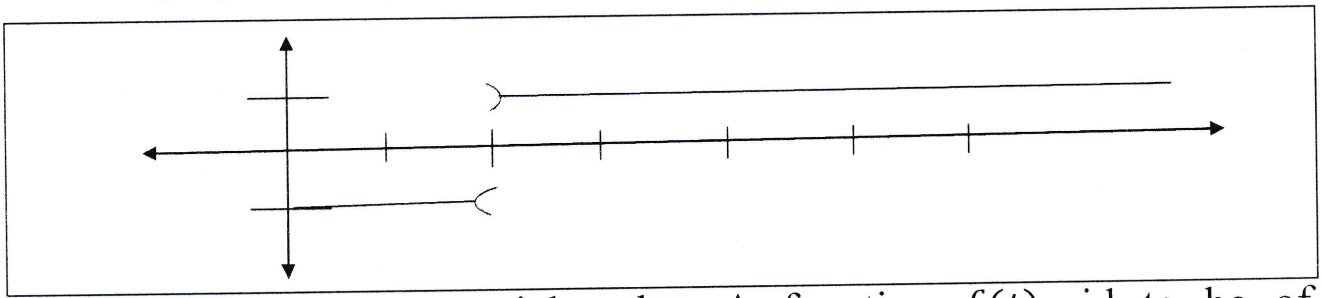
$$f(t) = \begin{cases} -1, 0 < t < 2 \\ 1\ , t > 2 \end{cases}$$

F is piecewise continuous on every finite interval $0 \leq t \leq b$, for every positive number b(>2).

At t=2 we have $f(2-) = \lim_{t \to 2-} f(t) = -1$

$$f(2+) = \lim_{t \to 2+} f(t) = +1$$

The graph of f(x) is shown in the figure



Functions of Exponential order: A function $f(t)$said to be of exponential order if there exists exist a constant α and a positive constants t and M such that

$e^{-a(t)}|f(t)| < M$ for all $t > t_0$ at which $f(t)$ is defined.

More explicitly, if f is of exponential order corresponding to some definite constant α, then we say that f is of exponential order e^{at}

Example:

Every bounded function is of exponential order, with constant $\alpha = 0$. Thus sin(bt) and cos(bt) are of exponential order.

Consider the function $f(t) = e^{at} \sin bt$ is of exponential order, α=a. $e^{-at}|f(t)| = e^{-at}e^{at}\ |\sin bt| = |\sin bt|$

Which is bounded for all t.

Also consider the function f(t) = tn , n > 0 then

$$e^{-at}|f(t)| = e^{-\alpha t}\ t^n .$$

For any $\alpha > 0$, $\lim\limits_{n\to\infty} e^{-\alpha t}t^n = 0$. Thus there exists M > 0 $t0 > 0$, such that $e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n < M$, for $t > t_0$

Hence $f(t) = t_n$ is not of exponential order, with the constant α equal to any positive number.

The function $f(t) = e^{t}$ is not of exponential order, $e^{-\alpha t}|f(t)| = e^{t^2-\alpha t}$ is unbounded as $t \to \infty$for all α.

Theorem : A comparison test for improper integral

(1) Let g and G be real function such that $0 \leq t \leq G(t)$ on $a \leq t \leq \infty$

(2) Suppose $\int_a^\infty G(t)$ on $a \leq t \leq \infty$

(3) Suppose g is integrable on every finite closed subinterval of $a \leq t < c\infty$.

Conclution : Then $\int_a^\infty g(t)dt$ exists.

Hypothesis : Let f be a real function that has the following properties.

(1) $f$is piecewise continuous in every finite closed interval $0 \leq t \leq b (b > 0)$.

(2) f is of exponential order, that is $\exists$ α, $M > 0$and $t_0 > 0$ s.t. $e^{-\alpha}|f(t)| < M$ for $t > t_0$.

Conclusion : The Laplace transform $\int_0^\infty e^{-st} f(t)$ of f exists for $s > \alpha$.

Function of Class A : A function $f(t)$ is piecewise continuous function on every finite interval in the range $t \geq 0$ and is of exponential order α as $t \to \infty$ then $f(t)$ is called function of classA.

Integral Transform:

An improper integral of the form

$$\int_{-\infty}^{\infty} K(s,t)F(t)dt$$

Is called integral of $F(t)$ if it is convergent.

Sometime it is denoted by $f(s)$ or $T(F(t))$.

Therefore $f(s) = T\{F(t)\} = \int_{-\infty}^{\infty} K(s,t)F(t)dt$

The Laplace Transform:

The function $K(s,\ t)$ appearing in the integral is called Kernel of the transformation. Here s is a parameter and dependent of $t,\ s$ may be real or complex number.

If we take $K(s,t) = f(x) = \begin{cases} e^{-s}, t \geq 0 \\ 0\ , t < 0 \end{cases}$

The above transformation become

$$f(s) = T\{F(t)\} = \int_0^{\infty} F(t)e^{-s}dt$$

This transform is known as Laplace transform.

Definition of Laplace Transform:

Let $f(t)$ be an arbiotrary function defined on the interval $0 \leq t \leq \infty$, then the Laplace transform of $f(t)$ denoted as $L\{f(t)\}$ or $\overline{f(s)}$, is defined as

$$L\{f(t)\} = \overline{f(s)} = \int_0^{\infty} e^{-st}f(t)\, dt\ .$$

Here L is called Laplace transform operator. The parameter s is real or complex number. In general the parameter s is taken to be a real positive number.

Theorem regarding existence of Laplace transform: If $f(t)$ is a function of class A, then Laplace transform of $f(t)$ or $L\{f(t)\}$ exists for all $s > \alpha$.

The Inversion Formula for the Laplace Transform:

If $F(s)$ is an analytic function of a complex variable s and is of order $O(p^{-k})$ is some half plane $Re(p) \geq \gamma$ where γ and k are real constants and $k > 1$, then the integral

$$\frac{1}{2\pi i}\int_{c-i\omega}^{c+i\omega} e^{sx}F(s)ds$$

Along any line $Re(p) = c > \gamma$ converges to a function $f(x)$ which is independent of c and whose Laplace transform is $F(s), Re(s) > \gamma$.

Furthermore, the function $f(x)$ is continuous for each $x \geq 0$ and is $O(e^{cx})$ as $x \to \infty$.

In the following section we will discuss some basic properties of Laplace transform and try to find behaviour of some elementary function under this transformation.

Linearity Properties of Laplace Transformation:

Let f_1 and f_2

Be two functions where Laplace transform exist, then

(1) $L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\}$

(2) $L\{\mu f(t)\} = \mu L\{f(t)\}$ where μ is a constant.

Combining (1) and (2) we can write

$$L\{(\mu_1 f_1(t) + \mu_2 f_2(t)\} = \mu_1 L\{f_1(t)\} + \mu_2 L\{f_2(t)\},$$

where μ_1 and μ_2 are arbitrary constants.

Example: Find $L\{(\sin at)^2\}$

Since $(\sin at)^2 = \frac{1}{2}(1 - \frac{1}{2}\cos 2at)$

We have, $L\{(\sin at)^2\} = L\{\frac{1}{2} - \frac{1}{2}\cos 2at\}$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2at\}$$

$L\{1\} = \frac{1}{s}$ and $L\{\cos 2at\} = \frac{s}{s^2+4a^2}$

Then $L\{(\sin at)^2\} = \frac{2a^2}{s(s^2+4a^2)}$

Laplace Transform of Some Elementary Function:

(1) Let $f(t) = c$ (constant)

Then $\bar{f}(s) = L\{f(t)\} = \int_0^t e^{-st}\, c dt = c \lim_{B\to\infty} e^{-st}$

$$= c \lim_{B\to\infty} \int_0^B e^{-st}\, dt$$

$$= \frac{c}{s} \lim_{B\to\infty} (1 - e^{-sB})$$

$\bar{f}(s) = \frac{c}{s}$ for $s > 0$.

(2) Let $f(t) = e^{at}$

Then $\overline{f}(s) = \int_0^\infty e^{-st}\, e^{at} dt$

$$= \lim_{B\to\infty} \int_0^B e^{-(s-a)t}\, dt$$

$$= \frac{1}{s-a}(1 - \lim_{B\to\infty} e^{-(s-a)t}$$

$$= \frac{1}{s-a} \text{ for } s > a$$

An interesting result can be obtained from here. If we take $f(t) = e^{iat}$ then $L\{e^{iat}\} = \bar{f}(s) = \frac{1}{s-ia}$

Which can be obtained from the previous result by replacing a by ia.

Thus $L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2}$

This implies $\Rightarrow L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$ (using linear property).

Equating real and imaginary parts from both sides ,

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

(3) Let $f(t) = t^n$, where n is a positive integer.

Then $\bar{f}(s) = \int_0^\infty e^{-st} t^n dt$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^n} du \text{ substituting } u = st$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u}\, u^n du$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad [\text{Since, } \Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx]$$

$$= \frac{n!}{s^{n+1}} \quad [\text{Since, } \Gamma(n+1) = n!\,]$$

Thus $\bar{f}(s) = \frac{n!}{s^{n+1}}$; $s > 0$.

Laplace Transform of Some Well Known Functions:

$f(t)$	$L\{f(t)\}$ or $\bar{f}(s)$
c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}$ where $s > a$
$\cos at$	$\frac{s}{s^2-a^2}$
$\sin at$	$\frac{a}{s^2-a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
t^n	$\frac{n!}{n+1}$ where $s > 0$
$t \sin at$	$\frac{2as}{(s^2-a^2)^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2-a^2)^2}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $b \in \mathbb{R}$

In the next section we will discuss Laplace Transform of a derivative of a function which is basic step for solving ordinary differential equation .

Laplace Transformation of a Derivative of a Function:

Let $f(t)$ be continuous and $f'(t)$ be piecewise continuous in $0 \leq t < T$ for all $T \to \infty$. Then $L\{f'(t)$ exist and it is given by $L\{f'(t)\} = sL\{f(t)\} - f(0)$.

Proof: Let us consider the integral

$$\int_0^T e^{-st} f'(t)dt$$

Integrating by parts, we get

$$\int_0^T e^{-st} f'(t)dt = e^{-st} f(T) - f(0) + s\int_0^T e^{-st} f(t)dt \quad \ldots\ldots (1)$$

As $f(t)$ is of exponential order α as $t \to \infty$, $|f(t)| \le Me^{\alpha T}$ for large t.

Thus for large T,

$$|e^{-sT} f(t)| = e^{-sT}|f(T)| \le e^{-sT} M e^{\alpha T}$$

i.e. $e^{-sT} f(T) \le Me^{-(s-\alpha)T} \to 0$ for $s > \alpha$.

Now making $T \to \infty$, in the equation (1), we have

$$\int_0^T e^{-st} f'(t)dt = 0 - f(0) + s\int_0^{\infty} e^{-st} f(t)dt$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

So the theorem is proved.

In general we can write

$$L\{f^n(t)\} = sL\{f^{n-1}(T)\} - f^{n-1}(0),$$ n being a positive integer.

Translation Property Of Laplace Transformation:

Let $f(t)$ be a function, Laplace transform exists for $s > \alpha$, then for any constant a,

$L\{e^{at} f(t)\} = f\ \ s - a)$, for $s > \alpha + a$.

Proof: $f(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

Replacing s by $(s - a)$, we have

$$f(s - a) = L\{f(t)\} = \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-st}\{e^{at} f(t)\}dt$$

$$= L\{e^{at}f(t)\}$$

Example: $L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2-b^2}$; $s > a$ since $L\{\sin bt\} = \frac{b}{s^2-b^2}$,$s > 0$.

The Convolution:

Let f, g be two function that are piecewise continuous on every finite closed interval $0 \leq t \leq 1$ and of exponential order.

The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^1 f(r)g(t - r)dr$$

Is called the convolution of the function f and g.

If the function f and g be piecewise continuous in every finite closed interval $0 \leq t \leq b$ and of exponential order e^{at}, then

$$L\{f * g\} = L\{f\}L\{g\} \text{ for } s > a.$$

Now we will solve an initial value problem by the help of the Laplace Transform.

Initial Value Problem:

Solve $\frac{d^2y}{dx^2} + y = 0$

Subject to the condition $y = 1$ and $\frac{dy}{dt} = 0$ when $t = 0$.

Solution: Clearly here y is a function of t

The given differential equation can be written as

$$y''(t) + y(t) = 0$$

Now we have from previous article ,

$L\{f'(t)\} = sL\{f(t)\} - f(0)$

Replacing f by f' we have,

$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$

Substituting the value of $L\{f'(t)\}$

$L\{f''(t)\} = s[sL\{f(t)\} - f(0)] - f'(0).$

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

So we can write

$$L\{y''(t)\} = s^2L\{y(t)\} - sy(0) - y'(0)$$

Our differential equation is

$$y''(t) - y(t) = 0$$

Taking Laplace transformation in both sides

$$L\{y''(t) + y(t)\} = L\{0\}$$

$$\Rightarrow L\{y''(t) + y(t)\} = 0$$

$$\Rightarrow s^2L\{y(t)\} - sy(0) - y(0) + L\{y(t)\} = 0$$

Substituting the values of $y(0)$ and $y'(0)$

$$(s^2 + 1)L\{y(t)\} - s = 0$$

$$\Rightarrow L\{y(t)\} = \frac{s}{s^2 + 1}$$

$$\Rightarrow L\{y(t)\} = L\{cost\}$$

$$\Rightarrow y(t) = cost$$

We can also solve boundary value problems.

Boundary Value Problems:

Solve $\frac{d^2y}{dt^2} + y = 0$

Subject to the condition $y(0) = 0$ and $y\left(\frac{\Pi}{2}\right) = 1$

Solution: Our differential equation is

$$y''(t) + y(t) = 0$$

Taking Laplace transformation of both sides,

$$L\{y''(t) + y(t)\} = L\{0\}$$

$\Rightarrow L\{y''(t) + y(t)\} = 0$, [using linearity property]

$$s^2L\{y(t)\} - sy(0) - y'(0) + L\{y(t)\} = 0$$

Here $y'(0)$ is not given, let $y'(0) = k$.

Substituting the values of $y(0)$ and $y'(0)$

$$(s^2 + 1)L\{y(t)\} - k = 0$$

$$\Rightarrow L\{y(t)\} = \frac{k}{s^2 + 1}$$

$$\Rightarrow L\{y(t)\} = kL\{sint\}$$

$$\Rightarrow L\{y(t)\} = L\{ksint\}$$

$$\Rightarrow y(t) = ksint$$

Now $y\left(\frac{\pi}{2}\right) = 1$, hence

$$1 = ksin\frac{\pi}{2} \Rightarrow k = 1$$

Thus $y(t) = sint.$

Solution of linear system of equation:

Use Laplace Transform to find solution of the system of ODE

$\frac{dy}{dx} - 6x + 3x = 8e^t \ldots\ldots\ldots\ldots (A)$

$$\Rightarrow \frac{dy}{dx} - 2x - y = 4e^t$$

That satisfies the initial condition

$$x(0) = -1$$

$$\Rightarrow y(0) = 0$$

Solution: Taking Laplace transform of both sides of each differential equation, we have

$$L\{x'\} - 6L\{x(t)\} + 3L\{y(t)\} = 8L\{e^t\} \ldots\ldots\ldots\ldots (B)$$

Now $L\{x'(t)\} = s\overline{x}(s) - x(0) = s\overline{x}(s) + 1$

$$\Rightarrow L\{y'(t)\} = s\overline{y}(s) - y(0) = s\overline{y}(s)$$

Also$L\{e^t\} = \frac{1}{s-1}$

Then equation (B) becomes

$$(s-6)\overline{x}(s) + 3\overline{y}(s) = \frac{8}{s-1} - 1$$

$$\Rightarrow -2\overline{x}(s) + (s-1)g(s) = \frac{4}{s-1}$$

$$i.e.\, (s-6)\overline{x}(s) + 3\overline{y}(s) = \frac{-s+9}{s-1} \;\ldots\ldots\ldots (C)$$

$$\Rightarrow -2\overline{x}(s) + (s-1)\overline{y}(s) = \frac{4}{s-1}$$

Now after solving the linear algebraic system of the two equations of two unknowns $\overline{x}(s)$ and $\overline{y}(s)$, we have

$$\overline{x}(s) = \frac{-s+7}{(s-1)(s-4)}$$

$$\overline{y}(s) = \frac{2}{(s-1)(s-4)}$$

Therefore $x(t) = L^{-1}\{\overline{x}(s)\} = L^{-1}\{\frac{-s+7}{(s-1)(s-4)}\}$

$$= L^{-1}\{\frac{-2}{s-1} + \frac{1}{s-4}\}$$

$$= -2L^{-1}\{\frac{1}{s-1}\} + L^{-1}\{\frac{1}{s-4}\}$$

$$= -2e^t + e^{4t}$$

And $y(t) = L^{-1}\{y(s)\} = L^{-1}\{\frac{2}{(s-1)(s-4)}\}$

$$= L^{-1}\{\frac{-2}{s-1} + \frac{1}{s-4}\}$$

$$= -2e^{t} + e^{4t}$$

And $y(t) = L^{-1}\{y(s)\} = L^{-1}\{\frac{2}{(s-1)(s-4)}\}$

$$= \frac{2}{3} L^{-1}\{\frac{1}{s-4}\} - \frac{2}{3} L^{-1}\{\frac{1}{s-1}\}$$

$$= \frac{2}{3}(e^{4t} - e^{t})$$

These are the required solution of the system of differential equation(A).

Conclusion: In this work introduce Laplace Transform. After that we state existence criteria of this transform then changes some elementary function through this transformation have been shown. Finally initial value problem, boundary value problem and system of linear differential equation with initial condition have been solved by this transformation.

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