LAPLACE TRANSFORM AND ITS APPLICATION TO SOLVE INITIAL AND BOUNDARY VALUE PROBLEMS

A project submitted by

Tushar Ghosh

B.Sc, VI semester, Mathematics Honours

Roll No: 190313700003, Reg. No: 201901020496

Of 2019-2020

Government General Degree College at Kalna-1(Kalna, Purbabardhaman)

Under guidance of

Dr. Dibakar Mondal

Assistant Professor, Department of Mathematics

Government General Degree College at Kalna-1

University of Burdwan

Acknowledgement

Primary I would thank God for being able to complete this project with success. Then I would like to thank our Principal sir Prof. (Dr.) Krishnendu Dutta, our departmental sir Dr. Mani Shankar Mandal(HOD), Dr. Dibakar Mondal and Prof. Tanmoy Mitra, Department of Mathematics of our College, They guide to complete the project.

Then I would like to thank my parents and friends who have helped me with this their valuable guidance has been helpful in various phases of the completion of this project.

Tushar Ghosh
(Tushar Ghosh)

Date: 11/06/2022

Contents

Introduction

In Mathematics, the Laplace transform is an integral transform is an Integral transform named after its inventor French Mathematician Pierre-Simon Laplace(1749-1827),and systematically developed by the British Physicist Oliver Heaviside(1850-1925),to simplify the solution of many differential equations that describe physical processes.

The technique of integral transforms is a powerful and indispensable tool for modern applied mathematics or theoretical physicists for successful investigation of boundary value problems arising in mathematical physics. Laplace or Fourier transforms are generally used to solve a boundary value problem where the governing partial differential equation is linear such as the Laplace's equation or the modified Helmholtz equation in the rectangular Cartesian co-ordinate system (cf,sneddon(1979)).Most of the integral transforms and their invertion formula are available in standard books on integral transform Erdelyiet al.(1954).

What we want to study in this project:

In this work we will describe the application of Laplace transformation in solving differential equation theoretically step by step manner. First we define it, then derive conditions under which it exists. After that we obtain Laplace transformation and inverse of some elementary function. Then we discuss Laplace transform of derivative of a function which is very important in this context. This will be followed by procedure of solving differential equation. Next we validate this procedure by discussion of a particular differential equation.

Piecewise continuous function:

A function $f(t)$ is said to be piecewise continuous function on a closed interval $a \leq t \leq b$ if the interval can be subdivided int finite number of intervals.in each of which $f(t)$ is continuous and has finite right and left hand limits.

Examples:

Consider the function f defined by

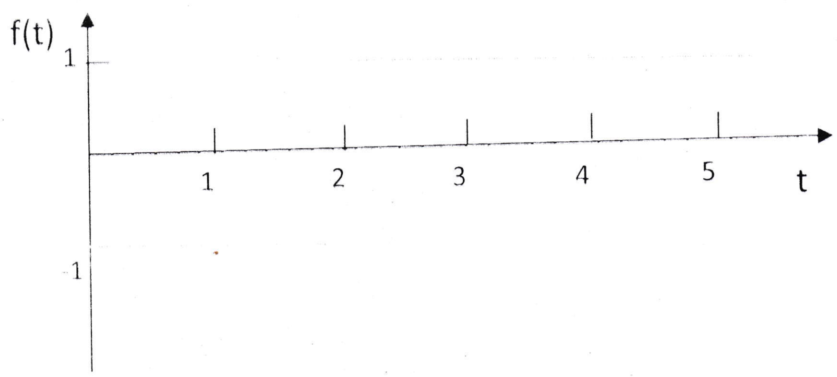
$$f(t) = \begin{cases} -1, & 0 < t < 2 \\ 1, & t > 2 \end{cases}$$

f is piecewise continuous on every finite interval $0 \leq t \leq b$, for every positive number $b(> 2)$.

At $t = 2$, we have $\qquad f(2^-) = \lim_{t\to 2^-} f(t) = -1$

$$f(2^+) = \lim_{t\to 2^+} f(t) = +1$$

The graph of $f(x)$is shown in the figure



Functions of Exponential order:

A function $f(t)$ is said to be exponential order if there exists a constant α and positive constants t_0 and M such that

$$e^{-\alpha t}|f(t)| < M \quad \text{for all } t > t_0 \text{ at which } f(t) \text{ is defined.}$$

More explicity, if f is of exponential order corresponding to some definite constant α, then we say that f is of exponential order $e^{\alpha t}$.

Example:

Every bounded function is of exponential order, with constant $\alpha = 0$. Thus $\sin(bt)$ and $\cos(bt)$ are of exponential order.

Consider the function $f(t) = e^{at}\sin bt$ is of exponential order, $\alpha = a$.

$$e^{-at}|f(t)| = e^{-at}e^{at}|\sin bt| = |\sin bt|$$

Which is bounded for all t.

Also consider the function $f(t) = t^n,\ n > 0$ then

$$e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n$$

For any $\alpha > 0$, $\lim_{n\to\infty} |e^{-\alpha t}t^n = 0$. Thus there exists $M > 0$ and $t_0 > 0$

Such that $e^{-\alpha t}|f(t)| = e^{-\alpha t}t^n < M,\ for\ t > t_0$.

Hence $f(t) = t^n$ is not of exponential order, with the constant α equal to any positive number.

The function $f(t) = e^{t^2}$ is not of exponential order, $e^{-\alpha t}|f(t)| = e^{t^2-\alpha t}$ is unbounded as $t \to \infty$ for all α.

Theorem:

A comparison test for improper integral:

1) Let g and G be real functions such that,

$$0 \leq g(t) \leq G(t) \text{ on } a \leq t < \infty$$

2) Suppose $\int_a^\infty G(t)dt$ exists.

3) Suppose g is integrable on every finite closed subinterval of $a \leq t < \infty$.

Conclusion:

Then $\int_a^\infty g(t)dt$ exists.

Theorem:

Hypothesis:

1) Suppose the real function g is integrable on every finite closed subinterval of $a \leq t \leq \infty$.

2) Suppose $\int_a^\infty |g(t)|dt$ exists.

Conclusion:

Then $\int_a^\infty g(t)dt$ exists.

Hypothesis:

Let f be a real function that has the following properties.

1) f is piecewise continuous in every finite closed interval $0 \leq t \leq b (b > 0)$.

2) f is of exponential order, that is,

$\exists\ \alpha, M > 0$ and $t_0 > 0$ s.t, $e^{-\alpha t}|f(t)| < M$ for $t > t_0$.

Conclution:

The Laplace transform $\int_0^{\infty} e^{-s t} f(t)$ of f exists for $s > \alpha$

Function of class A:

A function $f(t)$ is piecewise continuous function on every finite intervai in the range $t \geq 0$ and is of exponential order α as $t \to \infty$ then $f(t)$ is called function of class A.

Integral Transform:

An improper integral of the form

$$\int_{-\infty}^{+\infty} K(s,t)F(t)d\ t$$

is called integral transform of $F(t)$ if it is convergent. Sometime it is denoted by $f(s)$ or $T\big(F(t)\big)$. Therefore

$$f(s) = T\{F(t)\} = \int_{-\infty}^{+\infty} K(s,t)F(t)d\ t$$

The Laplace Transform:

The function $K(s,t)$ appearing in the integrand is call Kernal of the transformation. Here s is a parameter and dependent of t, s may be real or complex number.

If we take $K(\mathrm{s},\mathrm{t}) = f(x) = \begin{cases} e^{-s\ t}; & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then the above transformation become

$$f(s) = T\{F(t)\} = \int_0^{\infty} F(t)e^{-s\ t}d\ t$$

This transform is known as Laplace transform.

Definition of Laplace Transform:

Let $f(t)$ be an arbitrary function defined on the interval $0 \leq t < \infty$, then the Laplace transform of $f(t)$denoted as $L\{f(t)\}$ or $\overline{f(s)}$, is define as

$$L\{f(t)\} = \overline{f(s)} = \int_0^\infty e^{-st} f(t)dt.$$

Here L is called Laplace transform operator. The parameter s is real or complex number. In general the parameter s is taken to be a real positive number.

Theorem regarding existence of Laplace transforms:

If $f(t)$ is a function of clas A, then Laplace transform of $f(t)$ or $L\{f(t)\}$ exists for all $s > \alpha$.

The inversion Formula for the Laplace transform:

If $F(s)$ is an analytic function of the complex variable s and is of order $O(P^{-k})$ in some half plane $Re(P) \geq \gamma$ where γ *and* k are real constants and $k > 1$, then the integral

$$\frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{c-i\omega}^{c+i\omega} e^{sx} F(s) ds$$

Along any line $Re(p) = c > \gamma$ converges to a function $f(x)$ which is independent of c and whose Laplace transform is $F(s), Re(s) > \gamma$.

Furthermore, the function $f(x)$ is continuous for each $x \geq 0$ and is $O(e^{cx})$ *as* $x \to \infty$ (cf. Sneddon (1979)).

In the following section we will discuss some basis properties of Laplace transform and try to find behaviour of some elementary function under this transformation.

Linearity property of Laplace transform:

Let f_1 *and* f_2 be two functions whose Laplace transform exist, then

1) $L\{f_1(t) + f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\}$

2) $L\{\mu f(t)\} = \mu L\{f(t)\}$ where μ is a constant.

Combining (1) and (2), we can write

$L\{\mu_1 f_1(t) + \mu_2 f_2(t)\} = \mu_1 L\{f_1(t)\} + \mu_2 L\{f_2(t)\}$ where μ_1 *and* μ_2 are constants.

Example:

Find $L\{(\sin at)^2\}$

Since $(\sin at)^2 = \frac{1}{2}(1 - \cos 2at)$,

We have, $L\{(\sin at)^2\} = L\{\frac{1}{2} - \frac{1}{2}\cos 2at\}$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2at\}$$

$L\{1\} = \frac{1}{s}$ and $L\{\cos 2at\} = \frac{s}{s^2+4a^2}$

Then $L\{(\sin at)^2\} = \frac{2a^2}{s(s^2+4a^2)}$

Laplace transform of some elementary functions:

1) Let $f(t) = c$ (constant)

Then

$$\bar{f}(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} c\, dt = c \lim_{B \to \infty} \int_0^{B} e^{-st}\, dt$$

$$= c \lim_{B \to \infty} \int_0^{B} e^{-st}\, dt$$

$$= \frac{c}{s} \lim_{B \to \infty} (1 - e^{-sB})$$

$$\bar{f}(s) = \frac{c}{s} \; for \; s > 0$$

2) Let $f(t) = e^{at}$

Then

$$\bar{f}(s) = \int_0^{\infty} e^{-st} e^{at} dt$$

$$= \lim_{B \to \infty} \int_0^{B} e^{-(s-a)t} dt$$

$$= \frac{1}{s-a}(1 - \lim_{B \to \infty} e^{-(s-a)t})$$

$$= \frac{1}{s-a} \; for \;\; s > a.$$

An interesting result can be obtained from here. If we take $f(t) = e^{iat}$ then

$$L\{e^{iat}\} = \bar{f}(s) = \frac{1}{s-ia}$$

Which can be obtained previous result by replacing $a \; by \; ia$.

Thus $L\{\cos at + i \sin at\} = \frac{s+ia}{s^2+a^2}$

$$\Rightarrow L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$ (using linear property)

Equating real and imaginary parts from both sides,

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

3) Let $f(t) = t^n, where\ n\ is\ a\ positive\ integer.$

Then

$$\bar{f}(s) = \int_0^\infty e^{-st} t^n dt$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^n} \frac{1}{s} du \text{ substituting } u = st,\ s > 0$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \ [since\ \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx]$$

$$= \frac{n!}{s^{n+1}} \ [since\ \Gamma(n+1) = n!]$$

Thus $\bar{f}(s) = \frac{n!}{s^{n+1}}$ $s > 0$

Laplace transform of some well known functions:

$f(t)$	$L\{f(t)\}$ or $\bar{f}(s)$
c	c/s
e^{at}	$\frac{1}{s-a}$ where $s > a$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

Thus for large T,

$$|e^{-sT}f(t)| = e^{-sT}|f(T)| \leq e^{-sT}Me^{\alpha T}$$

$$\text{i.e., } e^{-sT}f(T) \leq Me^{-(s-\alpha)T} \to 0 \text{ for } s > \alpha$$

Now making $T \to \infty$, in the equation (1), we have

$$\int_0^T e^{-st}f'(t) = 0 - f(0) + s\int_0^\infty e^{-st}f(t)dt$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

So the theorem is proved.

In general we can write

$L\{f^n(t)\} = sL\{f^{n-1}(t)\} - f^{n-1}(0)$, n being a positive integer.

Translation property of Laplace transform:

Let $f(t)$ be a function, Laplace transform exists for $s > \alpha$, then for any constant a,

$$L\{e^{at}f(t)\} = \bar{f}(s-a), \text{ for } s > \alpha + a.$$

Proof: $\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

Replacing s by $(s-a)$, we have

$$\bar{f}(s-a) = L\{f(t)\} = \int_0^\infty e^{-(s-a)t}f(t)dt = \int_0^\infty e^{-st}\{e^{at}f(t)\}dt$$

$$= L\{e^{at}f(t)\}$$

Example:

$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2-b^2}$, $s > a$ since $L\{\sin bt\} = \frac{b}{s^2-b^2}$, $s > 0$.

The convolution:

Let f, g be two functions that are piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order. The function denoted by $f * g$ and defined by

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

is called the convolution of the functions f and g.

If the functions f and g be piecewise continuous on every finite closed interval $0 \leq t \leq b$ and of exponential order order e^{at},then

$L\{f * g\} = L\{f\}L\{g\}$ for $s > a$.

Now we will solve an initial value problem by the help of Laplace transform.

Initial value problem:

Solve $\frac{d^2y}{dx^2} + y = 0$

Subject to the condition $y = 1$ and $\frac{dy}{dt} = 0$ when $t = 0$.

Solution: Clearly here y is a function of t

The given differential equation can be written as

$$y''(t) + y(t) = 0$$

Now we have from previous article,

$$L\{f'(t)\} = sL\{f(t)\} - f(0).$$

Replacing f by f' we have,

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0).$$

Substituing the value of $L\{f'(t)\}$,

$$L\{f''(t)\} = s[sL\{f(t)\} - f(0)] - f'(0)$$

$$L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f'(0)$$

So we can write

$$L\{y''(t)\} = s^2L\{y(t)\} - sy(0) - y'(0).$$

Our differential equation is

$$y''(t) + y(t) = 0$$

Taking Laplace transform of sides,

$$L\{y''(t) + y(t)\} = L\{0\}$$

$$\Rightarrow L\{y''(t) + y(t)\} = 0 \text{ [using linearity property]}$$

$$\Rightarrow s^2L\{y(t)\} - sy(0) - y(0) + L\{y(t)\} = 0$$

Substituing the values of $y(0)$ and $y'(0)$

$$(s^2 + 1)L\{y(t)\} - s = 0$$

$$\Rightarrow L\{y(t)\} = \frac{s}{s^2+1}$$

$$\Rightarrow L\{y(t)\} = L\{\cos t\}$$

$$\Rightarrow y(t) = \cos t$$

We can also solve boundary value problems.

Boundary value problem:

Solve

$$\frac{d^2y}{dt^2} + y = 0$$

Subject to the condition $y(0) = 0$ and $y\left(\frac{\pi}{2}\right) = 1$.

Solution:

Our differential equation is

$$y''(t) + y(t) = 0$$

Taking Laplace transform of both sides,

$$L\{y''(t) + y(t)\} = L\{0\}$$

$$L\{y''(t) + y(t)\} = 0 \text{ [using linearity property]}$$

$$s^2L\{y(t)\} - sy(0) - y'(0) + L\{y(t)\} = 0$$

Here $y'(0)$ is not given, let $y'(0) = k$.

Substituing the values of $y(0)$ and $y'(0)$

$$(s^2 + 1)L\{y(t)\} - k = 0$$

$$L\{y(t)\} = \frac{k}{s^2+1}$$

$$L\{y(t)\} = kL\{\sin t\}$$

$$L\{y(t)\} = L\{k \sin t\}$$

$$y(t) = k \sin t$$

Now $y\left(\frac{\pi}{2}\right) = 1$, hence

$$1 = k \sin\frac{\pi}{2} \Rightarrow k = 1$$

Thus $y(t) = \sin t$

Solution of linear system of equation :

Use Laplace Transform to find solution of the system of ODE

$$\left.\begin{aligned} \frac{dx}{dt} - 6x + 3y = 8e^t \\ \frac{dy}{dt} - 2x - y = 4e^t \end{aligned}\right\} \text{---------} (A)$$

That satisfies the initial conditions

$$x(0) = -1$$

$$y(0) = 0.$$

Solution:

Taking Laplace transform of both sides of each differential equations, we have

$$\begin{aligned} L\{x'\} - 6L\{x(t)\} + 3L\{y(t)\} &= 8L\{e^t\} \\ L\{y'\} - 2L\{x(t)\} - L\{y(t)\} &= 4L\{e^t\} \end{aligned} \quad ---------(B)$$

Now

$$L\{x'(t)\} = s\bar{x}(s) - x(0) = s\bar{x}(s) + 1$$

$$L\{y'(t)\} = s\bar{y}(s) - y(0) = s\bar{y}(s)$$

Also

$$L\{e^t\} = \frac{1}{s-1}$$

Then equation (B) becomes

$$(s-6)\bar{x}(s) + 3\bar{y}(s) = \frac{8}{s-1} - 1$$

$$-2\bar{x}(s) + (s-1)\bar{y}(s) = \frac{4}{s-1}$$

i.e.

$$\begin{aligned} (s-6)\bar{x}(s) + 3\bar{y}(s) &= \frac{-s+9}{s-1} \\ -2\bar{x}(s) + (s-1)\bar{y}(s) &= \frac{4}{s-1} \end{aligned} \quad ---------(C)$$

Now after solving the linear algebraic system of the two equations (C) of two unknowns $\bar{x}(s)$ and $\bar{y}(s)$, we have

$$\bar{x}(s) = \frac{-s+7}{(s-1)(s-4)}$$

$$\bar{y}(s) = \frac{2}{(s-1)(s-4)}$$

Therefore

$$x(t) = L^{-1}\{\bar{x}(s)\} = L^{-1}\{\frac{-s+7}{(s-1)(s-4)}\}$$

$$= L^{-1}\{\frac{-2}{s-1} + \frac{1}{s-4}\}$$

$$= -2L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\{\frac{1}{s-4}\}$$

$$= -2e^{t} + e^{4t}$$

And

$$y(t) = L^{-1}\{\bar{y}(s)\} = L^{-1}\{\frac{2}{(s-1)(s-4)}\}$$

$$= \frac{2}{3}L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{2}{3}L^{-1}\{\frac{1}{s-1}\}$$

$$= \frac{2}{3}(e^{4t} - e^{t})$$

These are the required solution of the system of differential equations (A).

Conclution:

In this work we introduce Laplace transform. After that we state existence citeria of this transform. Then changes of some elementary function through this transformation have been shown. Finally initial value problem, boundary value problem and system of linear differential equation with initial condition have been solved by this transformation.

Bibligraphy:

1) Shepley L. Ross(2013), Differential Equation , Wiley

2) Schiff,J L.(1988), The Laplace Transform, theory and applications, Springer.

3) Spiegel, M. R. Laplace Transforms, schaum outlines.

4) Sneddon. I. N., The use of integral Transforms, Tata McGraw-Hill publication Co. Ltd.,New Delhi.1979.

5) Erdelyi,A.; Magnus, W.:Oberhettinger, F. And Tricomi, F., Tables of Integral Transforms, McGraw-Hill, Vols. 1 and 2, New York. 1954.